1. Electrostatics

The relevant Maxwell eqns are

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho & \left(\frac{\partial}{\partial t} \rho = 0\right) \\ \vec{\nabla} \times \vec{E} = 0 & \left(\frac{\partial}{\partial t} \vec{B} = 0\right) \end{cases}$$

Connection to Coulomb's law?

Force on a testcharge Q from pointcharge q located at $\vec{r}$ and $\vec{r}\,'$ respectively

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{Qq}{|\vec{r}-\vec{r}\,'|^3} (\vec{r}-\vec{r}\,')$$ based on exp!

"Convenient" to introduce vector $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ and define the electric field from q

$$\vec{F}(\vec{r}) = Q \underbrace{\frac{1}{4\pi\varepsilon_0} \frac{q}{|\vec{\mathscr{r}}|^3} \vec{\mathscr{r}}}_{\vec{E}(\vec{r})} = Q \frac{1}{4\pi\varepsilon_0} \frac{q}{\mathscr{r}^2} \hat{\mathscr{r}}$$

Generalise to several charges q_i at $\vec{r}_i'$

$$\vec{F}(\vec{r}) = Q \sum_i \frac{1}{4\pi\varepsilon_0} \frac{q_i}{|\vec{\mathscr{r}}_i|^3} \vec{\mathscr{r}}_i \qquad (\vec{\mathscr{r}}_i = \vec{r} - \vec{r}_i')$$

$$= Q \sum_i \vec{E}_i = Q \vec{E}(\vec{r})$$

based on exp!

note: force linear sum – superposition!

and to a continuous distribution $[dq = \rho\, d\tau']$

$$\vec{F} = Q \underbrace{\frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} (\vec{r}-\vec{r}\,')\, d\tau'}_{\vec{E}(\vec{r})}$$

integral over $\vec{r}\,'$

where ρ is the density. Three special cases: surface charge σ, line charge λ, and point charge q. Simple examples:

$$\rho(\vec{r}\,') = \sigma(x', y')\,\delta(z' - z_0)$$

$$\rho(\vec{r}\,') = \lambda(x')\,\delta(y' - y_0)\,\delta(z' - z_0)$$

$$\rho(\vec{r}\,') = q\,\delta^{(3)}(\vec{r}\,' - \vec{r}_0) =$$

$$= q\,\delta(x' - x_0)\,\delta(y' - y_0)\,\delta(z' - z_0)$$

Now let's take the divergence of $\vec{E}(\vec{r})$

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \left(\frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{(\vec{r}-\vec{r}')^3} (\vec{r}-\vec{r}')\, d\tau' \right)$$

$$= \left[\vec{\nabla} = \left(\hat{x}\frac{\partial}{\partial x}, \hat{y}\frac{\partial}{\partial y}, \hat{z}\frac{\partial}{\partial z}\right) \text{ only acts on } \vec{r} \right]$$

$$= \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}')\, \underbrace{\vec{\nabla} \cdot \left(\frac{\vec{r}-\vec{r}'}{(\vec{r}-\vec{r}')^3} \right)}_{4\pi \delta^{(3)}(\vec{r}-\vec{r}')} d\tau' = \frac{1}{\varepsilon_0} \rho(\vec{r}) \; !$$

Gauss's law can also be written on integral form using Gauss's theorem

$$\int_V \vec{\nabla} \cdot \vec{E}\, d\tau = \oint_S \vec{E} \cdot d\vec{S}$$

which together with $Q_{enc} = \int_V \rho(\vec{r})\, d\tau$ gives

$$\oint \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} Q_{enc}$$

Before leaving Coulomb's law we can also verify that $\vec{\nabla} \times \vec{E} = 0$. For simplicity we consider $\vec{E}(\vec{r})$ from a single charge at the origin $(\vec{r}' = 0)$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{\vec{r}^2} \hat{r}$$

Now take the curl and integrate over a surface S

$$\int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = \oint \vec{E} \cdot d\vec{l}$$

Stokes' theorem

using $d\vec{l} = \hat{r}\, dr + \hat{\theta}\, r\, d\theta + \hat{\varphi}\, r \sin\theta\, d\varphi$
finally gives

$$\oint \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0} \oint \frac{dr}{\vec{r}^{\,2}} = 0$$

$$\left[\int \frac{dr}{r^2} = -\frac{1}{r} \right]$$

true for arbitrary $S \Rightarrow \vec{\nabla} \times \vec{E} = 0$

superposition $\Rightarrow \vec{\nabla} \times \vec{E} = 0$ for any static charge distribution

—— x ——

Connection to field lines?

- start on positive and end on negative charges (or infinity)
- evenly spaced
- in all directions
- field strength given by (3d) density

mathematically these rules follow from

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho = \frac{1}{\varepsilon_0} q\, \delta^{(3)}(\vec{r} - \vec{r}\,') \text{ for point charge}$$

$$\vec{\nabla} \times \vec{E} = 0$$

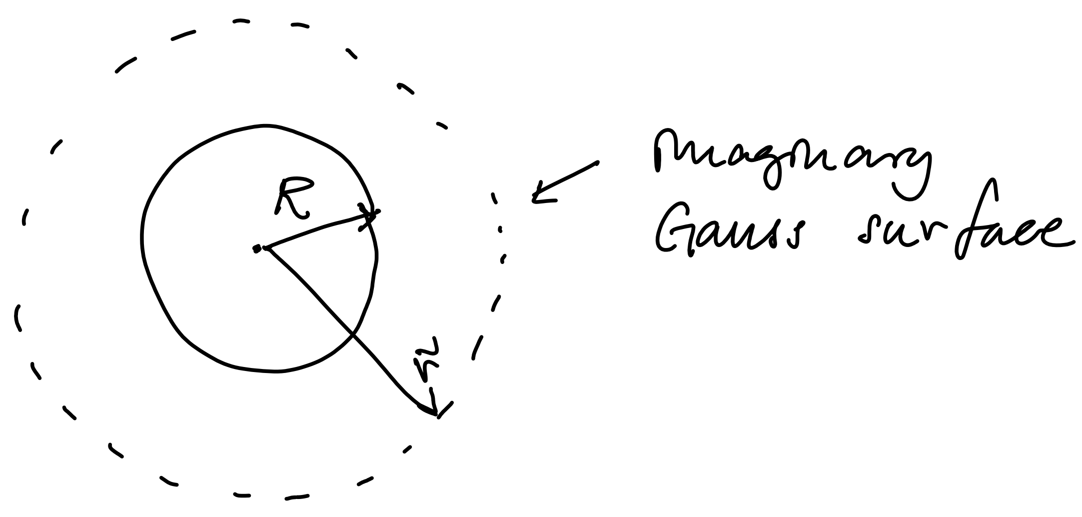
[cf chapt 1.2.4 and 1.2.5]

Example of (simple) application of Gauss's law:

What is the $\vec{E}$-field outside a uniformly charged solid sphere with radius R and total charge q?

Gauss's law on integral form:

$$\oint \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} Q_{enc}$$



1) choose spherical surface centered around the charge $d\vec{S} = \hat{r}\, dS_r = \hat{r}\, r^2 \sin\theta\, d\theta\, d\varphi$

2) symmetry dictates that $\vec{E}$ is directed radially outward $\vec{E} = \hat{r}\, E(r)$

3) $Q_{enc} = q$ (surface outside)

$$\Rightarrow \oint \vec{E} \cdot d\vec{S} = E \oint dS = E\, 4\pi r^2 = \frac{1}{\varepsilon_0} q$$

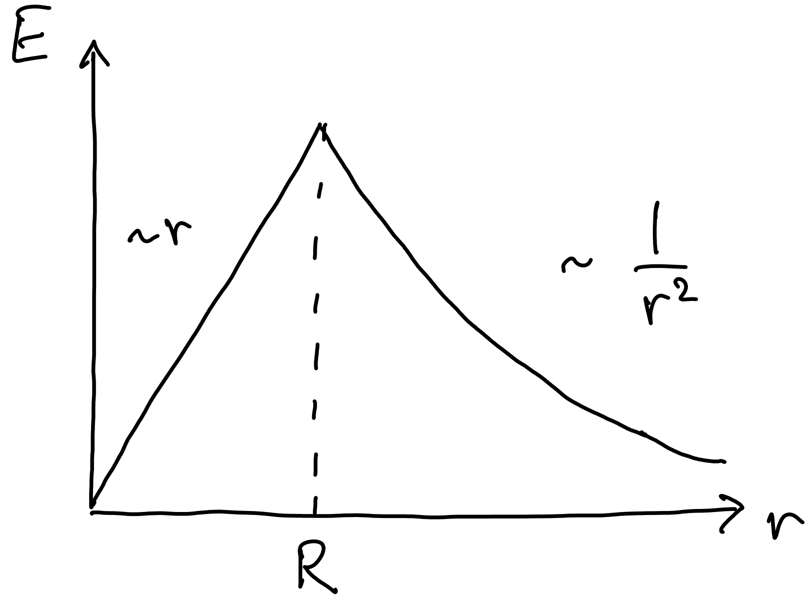
$$\therefore \quad \vec{E}_{outside} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

Can use the same method to calculate the field inside the sphere

3; $Q_{enc} = \int^{r} \rho \, d\tau' = \rho \int^{r} d\tau' = \rho \, \frac{4\pi r^3}{3}$

$\left[\rho = \frac{3q}{4\pi R^3}\right] = q \, \frac{r^3}{R^3}$

$\therefore \quad \vec{E}_{inside} = \frac{1}{4\pi\varepsilon_0} \, \frac{q r}{R^3} \, \hat{r}$



When we want to solve

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho \\ \vec{\nabla} \times \vec{E} = 0 \end{cases}$$

more generally it is convenient to introduce the "scalar" or electric potential, V

$$\vec{E} = -\vec{\nabla} V$$

↑ convention

or on integral form

$$V(\vec{r}) = -\int_{\vec{\mathcal{O}}}^{\vec{r}} \vec{E} \cdot d\vec{l}$$

where $\vec{\mathcal{O}}$ defines the boundary cond $V(\vec{\mathcal{O}}) = 0$.

This is possible since $\vec{\nabla} \times \vec{E} = 0$ and $\vec{\nabla} \times (\vec{\nabla} f) = 0$ for an arbitrary fcn f.

From the integral formulation it is also easy to see that V obeys the superposition principle (if $\vec{E}$ does)

Inserting $\vec{E} = -\vec{\nabla} V$ into Gauss's law gives

$$\vec{\nabla} \cdot (-\vec{\nabla} V) = \frac{1}{\varepsilon_0} \rho \quad \Rightarrow \quad \vec{\nabla}^2 V = -\frac{1}{\varepsilon_0} \rho$$

called Poisson's equation (we will come back to how to solve this more generally)

Start by finding out V for some charge distribution ρ. We know that

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}\,') \underbrace{\frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3}}_{-\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|} + \text{const}\right)} d\tau'$$

acts on $\vec{r}$

$$= -\vec{\nabla} \underbrace{\frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d\tau' + \cancelto{0}{\text{const}}}_{V(\vec{r})}$$

$$\therefore \; V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d\tau' + \cancelto{0}{\text{const}}$$

for a point charge at $\vec{r}_0\,'$: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\vec{r}-\vec{r}_0\,'|}$

The arbitrary integration constant is connected to the fact that V is not a physical observable – only differences in V can be measured.

To see this consider a set of charges q_i that produce a total field $\vec{E}$.

Now let's assume we want to add one more charge Q to the system.

There is a force on Q from the field

$$\vec{F} = Q\vec{E}$$

to move the charge from $\vec{a}$ to $\vec{b}$ we have to apply an opposite force $\vec{F} = -Q\vec{E}$ and the work done is

$$W = \int_{\vec{a}}^{\vec{b}} \vec{F} \cdot d\vec{l} = -Q \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l} =$$

$$= -Q \int_{\vec{a}}^{\vec{b}} (-\vec{\nabla} V) \cdot d\vec{l} = Q \left[V(\vec{b}) - V(\vec{a}) \right]$$

$$\therefore \quad V(\vec{b}) - V(\vec{a}) = \frac{W}{Q}$$

We can also use this to calculate the work (potential energy) needed to assemble a collection of charges from infinity

1) two charges q_1 and q_2 at $\vec{r}_1$ and $\vec{r}_2$

$$W = q_2 \frac{1}{4\pi\varepsilon_0} \frac{q_1}{|\vec{r}_2 - \vec{r}_1|}$$

[start with q_1 and bring in q_2 from inf.]

2) three charges

$$W = q_2 \frac{1}{4\pi\varepsilon_0} \frac{q_1}{|\vec{r}_2 - \vec{r}_1|} + q_3 \frac{1}{4\pi\varepsilon_0} \left(\frac{q_1}{|\vec{r}_3 - \vec{r}_1|} + \frac{q_2}{|\vec{r}_3 - \vec{r}_2|} \right)$$

3) n charges

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n-1} \sum_{j>i}^{n} \frac{q_i q_j}{|\vec{r}_j - \vec{r}_i|}$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{1}{2} \sum_{j \neq i}^{n} \frac{q_i q_j}{|\vec{r}_j - \vec{r}_i|}$$

$$= \frac{1}{2} \sum_{i=1}^{n} q_i \underbrace{\sum_{j \neq i}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_j}{|\vec{r}_i - \vec{r}_j|}}_{V(\vec{r}_i)\ \text{[from other charges]}} = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i)$$

4) continuous charge distr. [V from all charges]

$$W = \frac{1}{2} \int \rho(\vec{r}\,')\, V(\vec{r}\,')\, d\tau'$$

<u>note:</u> only valid if ρ, V limited for all $\vec{r}\,'$
so not for point charges
(gives infinite self-energy)

$$= \frac{1}{2} \int \varepsilon_0 \underbrace{\vec{\nabla}' \cdot \vec{E}(\vec{r}\,')\; V(\vec{r}\,')}_{\vec{\nabla}' \cdot (\vec{E}(\vec{r}\,') V(\vec{r}\,')) - \vec{E}(\vec{r}\,') \cdot \vec{\nabla}' V(\vec{r}\,')} d\tau' =$$

$$= \frac{1}{2} \varepsilon_0 \left[\underbrace{\oint \underset{\sim \frac{1}{r}}{V(\vec{r}\,')}\, \underset{\sim \frac{1}{r^2}}{\vec{E}(\vec{r}\,')} \cdot \underset{\sim r^2}{d\vec{S}}}_{\to 0 \text{ as } r \to \infty} - \int \vec{E}(\vec{r}\,') \cdot \underbrace{\vec{\nabla}' V(\vec{r}\,')}_{-\vec{E}(\vec{r}\,')} d\tau' \right]$$

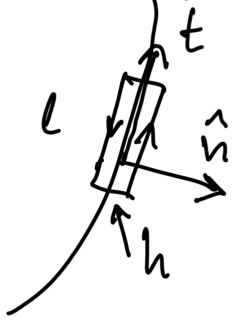
$$\therefore\; W = \int_{\text{all of space}} \underbrace{\frac{1}{2} \varepsilon_0 \vec{E}^2(\vec{r}\,')}_{u_E(\vec{r}\,'),\ \text{energy density}} d\tau'$$

[again this gives an infinite self-energy
of a point charge!]

Boundary conditions

Consider thin sheet with surface charge σ

1) inside outside



$\vec{E}_{out} \cdot \hat{t}$ for $\hat{t}$ in plane

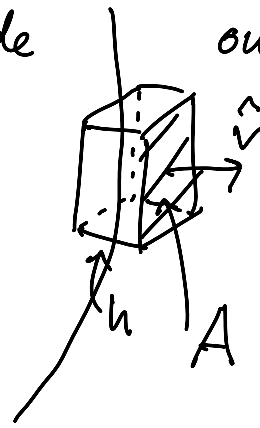
$$\oint \vec{E} \cdot d\vec{l} = E_{out}^{\parallel} \cdot l - E_{in}^{\parallel} \cdot l + O(h)$$

$$= (E_{out}^{\parallel} - E_{in}^{\parallel}) \cdot l = 0$$

$$[\vec{\nabla} \times \vec{E} = 0]$$

$$\Rightarrow \vec{E}_{out}^{\parallel} = \vec{E}_{in}^{\parallel}$$

2) inside outside



$\vec{E}_{out} \cdot \hat{n}$

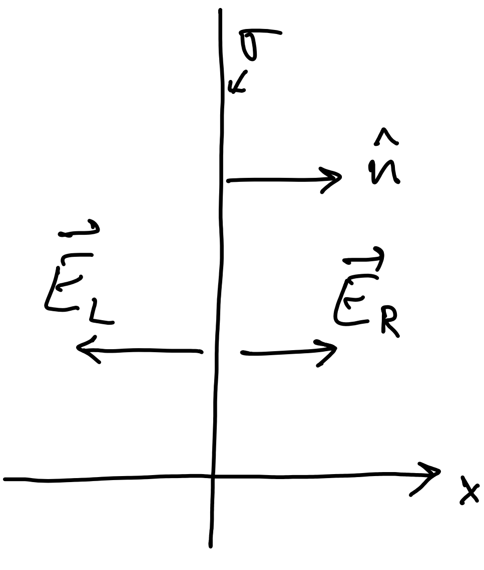
$$\oint \vec{E} \cdot d\vec{S} = E_{out}^{\perp} \cdot A - E_{in}^{\perp} \cdot A$$

$$= \frac{1}{\varepsilon_0} \sigma A$$

$$\Rightarrow E_{out}^{\perp} - E_{in}^{\perp} = \frac{1}{\varepsilon_0} \sigma$$

$$1) + 2) \Rightarrow \vec{E}_{out} - \vec{E}_{in} = \frac{1}{\varepsilon_0} \sigma \hat{n}$$

For an infinite plane with uniform surface charge σ this gives



symmetry: $\vec{E}_R = -\vec{E}_L = E(x)\hat{n}$

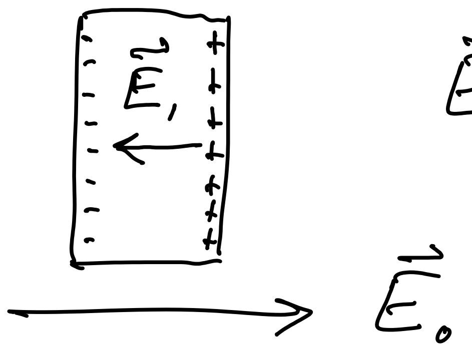
$$\left.\begin{array}{l} \oint \vec{E} \cdot d\vec{S} = 2E(x)A \\ Q_{enc} = A\sigma \end{array}\right\} E(x) = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

independent of distance!

Conductors

In a perfect conductor the electrons can move completely freely – gives several consequences: valid in electrostatic case!

a) $\vec{E} = 0$ inside – charges move until field vanishes



$\vec{E}_1 + \vec{E}_0 = 0$ inside

b) as a consequence $\rho_{inside} = \varepsilon_0 \vec{\nabla} \cdot \vec{E} = 0$

c) can have non-zero surface charge σ

d) the potential $V = -\int_{\vec{0}}^{\vec{r}} \vec{E} \cdot d\vec{l} = $ const inside (equipotential)

e) $\vec{E} \parallel \hat{n}$ just outside conductor

look also at boundary conditions

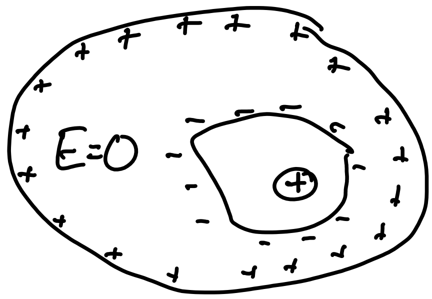
$$\left.\begin{array}{l} E_{in}^{\parallel} = 0 \Rightarrow E_{out}^{\parallel} = 0 \\ E_{in}^{\perp} = 0 \Rightarrow E_{out}^{\perp} = \frac{1}{\varepsilon_0}\sigma \end{array}\right\} \quad \vec{E}_{out} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

(just outside surface)

Induced charges

external field polarises conductor to keep
$\vec{E} = 0$ inside
$\Rightarrow \rho$ modified in non-trivial way

Ex. uncharged conductor
with point-charge q
inside cavity

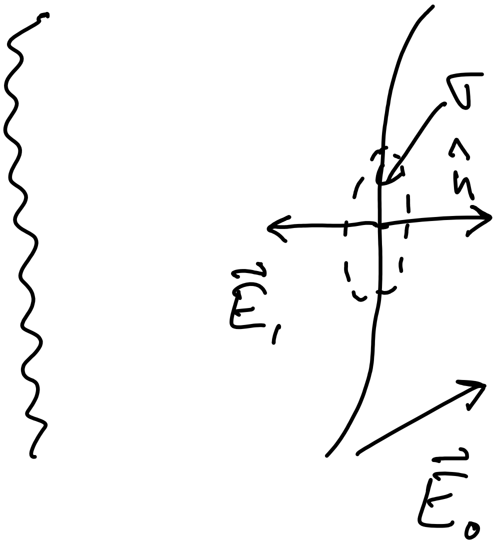


Field outside same as
from conductor charged with q –
irrespective of shape and position
of cavity and position of point charge

Force on a conductor in external field

Consider small patch on surface with σ and $\hat{n}$ constant in an "external" field $\vec{E}_0$

Gauss law $\Rightarrow$ field from patch is $\pm \vec{E}_1 = \mp \frac{1}{2}\frac{\sigma}{\varepsilon_0}\hat{n}$



for cond. σ is such that

$$\vec{E}_{in} \overset{\text{general}}{=} \vec{E}_1 + \vec{E}_0 = 0$$

$$\vec{E}_{out} = -\vec{E}_1 + \vec{E}_0 = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$\Rightarrow \vec{E}_0 \underset{\text{general}}{=} \frac{1}{2}\left(\vec{E}_{in} + \vec{E}_{out}\right) \equiv \vec{E}_{ave} \underset{\text{cond.}}{=} \frac{1}{2}\frac{\sigma}{\varepsilon_0}\hat{n}$$

$\Rightarrow$ net force on conductor per surface area from field except the patch considered

$$\frac{\vec{F}}{A} \underset{\text{general}}{=} \sigma \vec{E}_{ave} \underset{\text{cond}}{=} \frac{1}{2}\frac{\sigma^2}{\varepsilon_0}\hat{n} = p\,\hat{n}$$

electrostatic pressure

$$p = \frac{1}{2}\frac{\sigma^2}{\varepsilon_0} = \frac{1}{2}\varepsilon_0 E_{out}^2$$

Capacitors

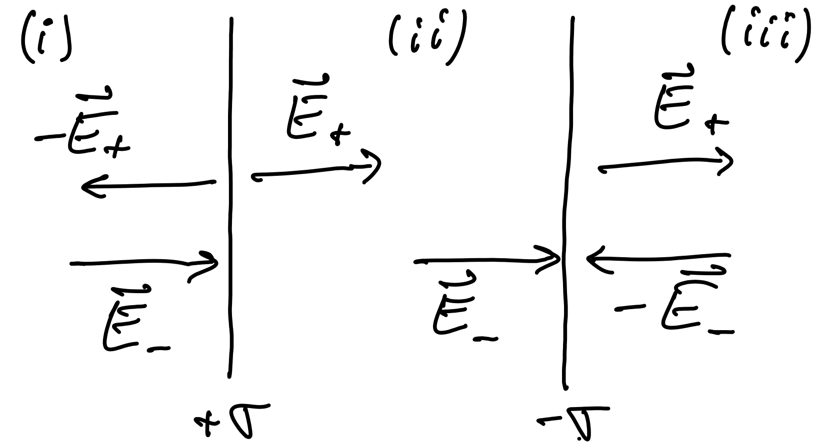
Consider two conductors with charges $\pm Q$ and potential difference $V = V_+ - V_-$ then

$$C \equiv \frac{Q}{V}$$

is the capacitance

[the pot. diff. is prop to the charge, "same" charge on both cond]

Example: two oppositely charged "infinite" planes of perfect conductor with uniform surface charges $\pm\sigma$, $\sigma = \frac{Q}{A}$



$$E_\pm = \frac{\sigma}{2\varepsilon_0}$$

region (i) $\vec{E} = \vec{E}_- - \vec{E}_+ = 0$

" (ii) $\vec{E} = \vec{E}_- + \vec{E}_+ = \frac{\sigma}{\varepsilon_0}$

" (iii) $\vec{E} = -\vec{E}_- + \vec{E}_+ = 0$

The potential difference

$$V = -\int_{(-)}^{(+)} \vec{E} \cdot \vec{dl} = |\vec{E}|\, d = \frac{\sigma d}{\varepsilon_0} = \frac{Q d}{A \varepsilon_0}$$

opposite direction to $\vec{E}$

$$\Rightarrow C = \frac{A\varepsilon_0}{d}$$

Energy stored in capacitor?

Consider a capacitor with potential difference V' and charge $Q' = CV'$

Making an infinitesimal increase of the charge, $Q' \to Q' + dQ'$ the work done is

$$dW = V'dQ' = \frac{1}{C} Q'dQ'$$

integrating from 0 to Q gives

$$W = \int_0^Q \frac{1}{C} Q'dQ' = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} CV^2$$

note: C depends on geometry - is constant when charging a capacitor but V and Q changes (in proportion)