

Homework problems

Group II

Problem No 1.9

Ultra-high energy neutrinos in the Universe

If there exist neutrinos of ultra-high energies in Nature, they may scatter off relic neutrinos. The neutrino-neutrino cross section in the Standard Model of particle physics is very small. Its maximum value, $\sigma = 0.15 \mu b = 1.5 \cdot 10^{31} \text{ cm}^2$, is reached at center-of-mass energy $\sqrt{s} \approx M_Z \approx 90 \text{ GeV}$ when neutrinos pair-annihilate through the resonant Z-boson production. The observation of photons produced in Z-decay chains would be indirect evidence for the existence of relic neutrinos. Find the mean free path of ultra-high energy neutrino in the present Universe with respect to the above process. Does one expect a cutoff in the spectrum of ultra-high energy neutrinos, similar to the GZK cutoff in the spectrum of ultra-high energy protons (see Problem 1.8)?

Problem A5

useful GR math

Check the following properties of the Christoffel symbols and covariant derivative:

$$\begin{aligned}\Gamma_{\nu\mu}^{\mu} &= \partial_{\nu} \ln \sqrt{-g}, \\ g^{\mu\nu} \Gamma_{\mu\nu}^{\lambda} &= -\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\lambda\mu}), \\ \nabla_{\mu} A^{\mu} &= \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} A^{\mu}), \\ \nabla_{\mu} A^{\mu\nu} &= \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} A^{\mu\nu}), \quad \text{for } A^{\mu\nu} = -A^{\nu\mu}, \\ \nabla_{\mu} \nabla^{\mu} \phi &= \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \partial_{\nu} \phi),\end{aligned}$$

Problem No 2.6

Klein-Gordon equation for a scalar field in expanding Universe

Show that for a slowly varying scale factor, solutions to equation

$$\frac{1}{a^2} \partial_{\eta} (a^2 \partial_{\eta} \phi) - \partial_i \partial_i + m^2 a^2 \phi = 0$$

are superpositions of

$$\phi(\eta, \vec{x}) = \frac{1}{a(\eta) \sqrt{\Omega(\eta)}} e^{i \int^{\eta} \Omega(\eta) d\eta} e^{-i \vec{k} \cdot \vec{x}} \cdot (1 + \mathcal{O}(\partial_{\eta} a)),$$

where $\Omega(\eta) = \sqrt{k^2 + m^2 a^2(\eta)}$. Thus, coordinate frequency (derivative of the exponent with respect to conformal time η) equal $\Omega(\eta)$, while the physical frequency is

$$\omega(\eta) = \frac{\Omega(\eta)}{a(\eta)}$$

Problem 3.7

Speed of sound in a universe with barotropic e.o.s.

Is it possible in expanding Universe with equation of state $p = p(\rho)$ to evolve from the regime $(p + \rho) > 0$ to the regime $(p + \rho) < 0$ without violating the real-valuedness of the sound speed us defined as $u_s^2 = \partial p / \partial \rho$?