Written exam problems, part 1

Problem No 2.2

By direct calculation, obtain the relations for 3-tensors

$$ {}^{(3)}R_{ijkl} = \frac{\kappa}{R^2}(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \,, \qquad {}^{(3)}R_{ij} = 2\frac{\kappa}{R^2}\gamma_{ij} $$

where γ_{ij} is the 3-metric and $\kappa = 0, \pm 1$ corresponding to three possible types of homogeneous and isotropic universes.

Problem No 3.2

Prove that equations

$$ R_{0i} - \frac{1}{2}g_{0i}R = 8\pi G T_{0i}, \quad R_{ij} - \frac{1}{2}g_{ij}R = 8\pi G T_{ij}, \quad \nabla_\mu T^{\mu i} = 0 $$

are identically satisfied in the case of homogeneous and isotropic Universe, provided that the following equations are satisfied,

$$ R_{00} - \frac{1}{2}g_{00}R = 8\pi G T_{00}, \quad \nabla_\mu T^{\mu 0} = 0 \,. $$

Do not assume in the proof that the Universe is filled with ideal fluid; it is important only that matter is homogeneous and isotropic.

Problem No 3.11

Consider the Universe filled with matter whose equation of state is that of Chaplygin gas, namely,

$$ p = -\frac{A}{\rho} \,, \quad A = const \,, $$

- (1) Find the dependence of the Hubble parameter on the scale factor.
- (2) Find the law of evolution $a = a(t)$ at small and large scale factors in all three cases with $\varkappa = 0, \pm 1$.
- (3) Find the complete evolution $a = a(t)$ in the case of spatially flat Universe.
- (4) What values of $\varkappa$ admit static solutions to the Einstein equations?
- (5) What can be said about the future of the Universe, if it is known that at some moment of time the expansion of the Universe accelerates? Consider all three cases with $\varkappa = 0, \pm 1$.

Problem No 4.3

At what z does the transition from deceleration to acceleration occur for dark energy with equation of state $p = w\rho$, $w = const$? For what value of the parameter w this transition would occur now? Give numerical estimate using the values

$$\Omega_M \approx 0.27 \quad \Omega_\Lambda \approx 0.73\,.$$

Problem No 4.6

Find the present age of the Universe for dark energy with equation of state $p = w\rho$, $w = const$. Give numerical estimates for $w = -1.1$ and $w = -0.9$ with $\Omega_M = 0.27$, $\Omega_\Lambda = 0.73$.