Lecture II

<u>What do I need to know about General Relativity?</u>

Lorentz-invariant interval:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

Coordinate transformation

$$x^\mu \to x'^\mu(x^\mu)$$

Transformation of a Lorentz scalar:

$$\phi'(x') = \phi(x)$$

Transformation of a Lorentz vector:

- contravariant

$$A'^\nu(x') = \frac{\partial x'^\nu}{\partial x^\mu}A^\mu(x)$$

- covariant

$$A'_\nu(x') = \frac{\partial x^\mu}{\partial x'^\nu}A_\mu(x)$$

Metric tensor transformation

$$g'_{\mu\nu}(x') = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu}g_{\lambda\rho}(x)$$

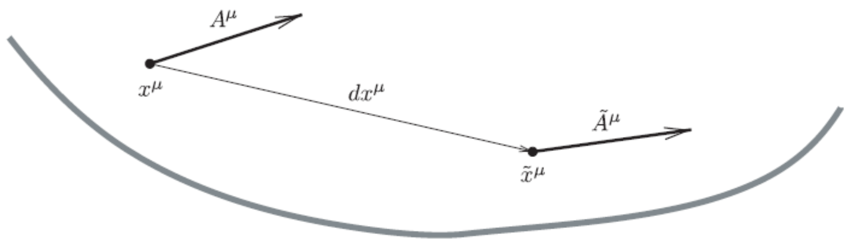
Determinant of the metric tensor (<0)

$$g \equiv \det g_{\mu\nu}$$

Lorentz-invariant 4D volume

$$\sqrt{-g}d^4x$$

Parallel transport of a Lorentz vector



Transformation of a vector under the parallel transport

$$\tilde{A}^\mu(\tilde{x}) = A^\mu(x) - \Gamma^\mu_{\nu\lambda}(x) A^\nu(x) dx^\lambda$$

Transformation of connection coefficients under the parallel transport (is NOT a tensor!)

$$\Gamma'^\mu_{\nu\lambda}(x') = \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\lambda}\frac{\partial x'^\mu}{\partial x^\xi}\Gamma^\xi_{\rho\sigma} + \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\nu \partial x'^\lambda}$$

Covariant derivative – generalization of ordinary derivative in a curved space-time:

$$A^\mu(\tilde{x}) - \tilde{A}^\mu(\tilde{x}) = \nabla_\nu A^\mu \cdot dx^\nu$$

Covariant derivatives of Lorentz tensors

$$\nabla_\nu A^\mu(x) = \partial_\nu A^\mu(x) + \Gamma^\mu_{\lambda\nu} A^\lambda(x)$$

$$\nabla_\nu B_\mu(x) = \partial_\nu B_\mu(x) - \Gamma^\lambda_{\mu\nu} B_\lambda(x)$$

$$\nabla_\mu B^\nu_{\lambda\tau} = \partial_\mu B^\nu_{\lambda\tau} + \Gamma^\nu_{\rho\mu} B^\rho_{\lambda\tau} - \Gamma^\rho_{\lambda\mu} B^\nu_{\rho\tau} - \Gamma^\rho_{\tau\mu} B^\nu_{\lambda\rho}$$

Riemannian geometry (manifold) conditions – the geometry of the Universe is Riemannian

- commutation of metric and covariant derivative (metric connection)

$$g_{\mu\nu} \nabla_\lambda A^\nu = \nabla_\lambda (g_{\mu\nu} A^\nu)$$

or

$$\nabla_\mu g_{\nu\lambda} = 0$$

- torsionless metric connection

$$C^\lambda_{\mu\nu} \equiv \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 0$$

Connection coefficients in Riemannian geometry – Christoffel symbols (is NOT a tensor!)

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho}(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda})$$

Useful formulas:

$$\Gamma^\mu_{\nu\mu} = \partial_\nu \ln \sqrt{-g},$$

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = -\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g} g^{\lambda\mu}),$$

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g} A^\mu),$$

For an antisymmetric tensor

$$\nabla_\mu A^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}A^{\mu\nu})$$

For a scalar

$$\nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)$$

Generalized Gauss Law (partial differentiation rule)

$$\int(\nabla_\nu A^\nu)\sqrt{-g}d^4x = \int \partial_\nu(\sqrt{-g}A^\nu)d^4x = \int \sqrt{-g}A^\nu d\Sigma_\nu$$

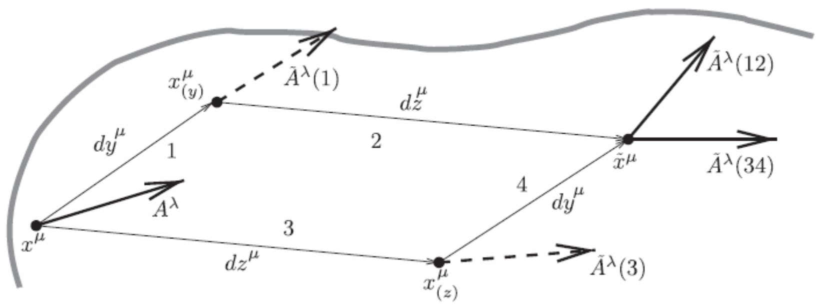
Major curvature characteristics – the Riemann tensor – the result of action of the commutator on a Lorentz vector

$$\nabla_\mu\nabla_\nu A^\lambda - \nabla_\nu\nabla_\mu A^\lambda = A^\sigma R^\lambda_{\sigma\mu\nu}$$

Expression through Christoffel symbols

$$R^\mu_{\nu\lambda\rho} = \partial_\lambda\Gamma^\mu_{\nu\rho} - \partial_\rho\Gamma^\mu_{\nu\lambda} + \Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\nu\rho} - \Gamma^\mu_{\sigma\rho}\Gamma^\sigma_{\nu\lambda}$$

The physical meaning of the Riemannian tensor – parameterizes a change of a Lorentz vector under parallel transport along different paths



$$\tilde{A}^\lambda(12) - \tilde{A}^\lambda(34) = A^\sigma R^\lambda_{\sigma\mu\nu}dz^\mu dy^\nu$$

Ricci tensor

$$R_{\mu\nu} \equiv R^\lambda_{\mu\lambda\nu}$$
$$R_{\mu\nu} = \partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\mu\Gamma^\lambda_{\lambda\nu} + \Gamma^\lambda_{\rho\lambda}\Gamma^\rho_{\mu\nu} - \Gamma^\lambda_{\rho\mu}\Gamma^\rho_{\nu\lambda}$$

Ricci scalar or scalar curvature

$$R \equiv g^{\mu\nu}R_{\mu\nu}$$

Properties of the Riemann tensor:

- symmetric under permutation of pairs of indices
- antisymmetric in the first and second pair of indices
- sum of permutations of three indices vanish

$$R_{\rho\mu\nu\lambda} + R_{\rho\lambda\mu\nu} + R_{\rho\nu\lambda\mu} = 0$$

- Bianchi identity

$$\nabla_\rho R^\lambda_{\sigma\mu\nu} + \nabla_\nu R^\lambda_{\sigma\rho\mu} + \nabla_\mu R^\lambda_{\sigma\nu\rho} = 0$$

Cosmological constant term in the action

$$S_\Lambda = -\Lambda \int d^4x\sqrt{-g}$$

Einstein-Hilbert action for the gravitational field

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R$$

Total action for the gravitational field

$$S_{gr} = S_\Lambda + S_{EH}$$

Variations of a matrix and a determinant of metric

$$\det(M+\delta M) = \det(M)\left[1 + \mathrm{Tr}(M^{-1}\delta M) + o(\delta M)\right]$$

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu}$$

$$\delta g^{\mu\nu} = -g^{\mu\rho}\delta g_{\rho\lambda}g^{\lambda\nu}$$ (is NOT a tensor!)

Hamilton's principle (or *principle of the least action* – variation of the action along the physical path vanishes)

Variation of the cosmological constant term

$$\delta S_\Lambda = -\Lambda\int d^4x\,\delta(\sqrt{-g}) = -\frac{\Lambda}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}$$

Variation of the Einstein-Hilbert contribution

$$\delta S_{EH} = \delta S_1 + \delta S_2 + \delta S_3$$

where

$$\delta S_1 = -\frac{1}{16\pi G}\int d^4x\, R\,\delta(\sqrt{-g}),$$

$$\delta S_2 = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}$$

$$\delta S_3 = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}$$

A straightforward calculation (your home-work No I) gives

$$\delta S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\delta g_{\mu\nu}$$

Einstein equations (without matter included!)

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G\Lambda g^{\mu\nu}$$

Matter term (Lagrangian of fields/particles in curved space-time)

$$S_m = \int d^4x\sqrt{-g}\mathcal{L}_m$$

Energy-momentum tensor definition

$$\delta S_m = \frac{1}{2}\int d^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}$$

Einstein equations with matter

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(\Lambda g_{\mu\nu} + T_{\mu\nu})$$

Example I: scalar fields in curved space-time

$$S_{sc} = \int d^4x\sqrt{-g}\mathcal{L}_{sc} = \int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right)$$

$$T^{sc}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}_{sc}$$

Example II: electromagnetic field

$$S_{em} = -\frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F_{\lambda\rho}g^{\mu\lambda}g^{\nu\rho}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$T^{em}_{\mu\nu} = -F_{\mu\lambda}F_{\nu\rho}g^{\lambda\rho} + \frac{1}{4}g_{\mu\nu}F_{\lambda\rho}F^{\lambda\rho}$$

Covariant conservation law (to prove this!):

$$\nabla_\mu T^{\mu\nu} = 0.$$

No conserved energy and momentum can be defined in curved space-time!

<u>What do I need to know about the Standard Model of particle physics?</u>

Particle content

(a) gauge bosons: photon, gluon, $W^\pm$-bosons, Z-boson;
(b) quarks: u, d, s, c, b and t;
(c) leptons: electrically charged (electron e, muon μ and τ-lepton) and neutral (electron neutrino ν_e, muon neutrino ν_μ and τ-neutrino ν_τ);
(d) neutral Higgs boson h.

Standard Model parameters:

$$\begin{array}{lll} m_e = 0.511 \text{ MeV}, & m_u = 1.5 - 3.3 \text{ MeV}, & m_d = 3.5 - 6.0 \text{ MeV}, \\ m_\mu = 105.7 \text{ MeV}, & m_c = 1.14 - 1.34 \text{ GeV}, & m_s = 0.07 - 0.13 \text{ GeV}, \\ m_\tau = 1.78 \text{ GeV}, & m_t = 169.1 - 173.3 \text{ GeV}, & m_b = 4.13 - 4.37 \text{ GeV}, \\ M_Z = 91.2 \text{ GeV}, & M_W = 80.4 \text{ GeV}, & v = 247 \text{ GeV}, \\ \alpha \equiv \dfrac{e^2}{4\pi} = \dfrac{1}{137}, & \sin^2\theta_W = 0.231, & \alpha_s(M_Z) = 0.118. \end{array}$$

Structure of gauge sector

$$SU(3)_c \times SU(2)_W \times U(1)_Y$$

Structure of fermion sector (three generations)

$$\begin{array}{rl} \text{I}: & u,\ d,\ \nu_e,\ e \\ \text{II}: & c,\ s,\ \nu_\mu,\ \mu \\ \text{III}: & t,\ b,\ \nu_\tau,\ \tau. \end{array}$$

The latter are spinor representations of the full Lorentz group

Dirac spinor in terms of Weyl spinors (“left” and “right” eigenfunctions of helicity operator with eigenvalues –1 and +1, respectively)

$$\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

Dirac equation

$$i\gamma^\mu \partial_\mu \psi = m\psi$$

$$\begin{pmatrix} 0 & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & 0 \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = m \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

Chiral projections

$$P_\mp = \frac{1 \mp \gamma^5}{2}, \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3. \quad \psi_L \equiv P_-\psi = \frac{1-\gamma_5}{2}\psi, \quad \psi_R \equiv P_+\psi = \frac{1+\gamma_5}{2}\psi$$

Lorentz structures in Lagrangian

$$\begin{array}{llll} \bar{\psi}\psi, & \text{scalar}, & \bar{\psi}\gamma^\mu\psi, & \text{vector}, \\ \bar{\psi}\gamma^5\psi, & \text{pseudoscalar}, & \bar{\psi}\gamma^5\gamma^\mu\psi, & \text{pseudovector} \end{array}$$
$$\bar{\psi} \equiv \psi^\dagger\gamma^0$$

SU(2)w vector/scalar representations

$$Q_1 = \begin{pmatrix} u \\ d \end{pmatrix}_L, \quad Q_2 = \begin{pmatrix} c \\ s \end{pmatrix}_L, \quad Q_3 = \begin{pmatrix} t \\ b \end{pmatrix}_L,$$
$$L_1 = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad L_2 = \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad L_3 = \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L.$$
$$U_n = u_R,\ c_R,\ t_R\ ; \quad n = 1, 2, 3\ ;$$
$$D_n = d_R,\ s_R,\ b_R\ ;$$
$$E_n = e_R,\ \mu_R,\ \tau_R.$$

Quantum numbers

Field\Group	$SU(3)_c$	$SU(2)_W$	$U(1)_Y$	$U(1)_{em}$
$L \equiv \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	1	2	-1	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
$E \equiv e_R$	1	1	-2	-1
$Q \equiv \begin{pmatrix} u \\ d \end{pmatrix}_L$	3	2	$+1/3$	$\begin{pmatrix} +2/3 \\ -1/3 \end{pmatrix}$
$U \equiv u_R$	3	1	$+4/3$	$+2/3$
$D \equiv d_R$	3	1	$-2/3$	$-1/3$

Covariant derivatives

$$\mathcal{D}_\mu f \equiv \left(\partial_\mu - ig_s T_s^a G_\mu^a - igT_W^i V_\mu^i - ig'\frac{Y_f}{2}B_\mu\right) f$$

$$\mathcal{D}_\mu H = \left(\partial_\mu - ig\frac{\tau^i}{2}V^i_\mu - i\frac{g'}{2}B_\mu\right)H$$

Standard Model Lagrangian

$$\begin{aligned}\mathcal{L}_{SM} &= -\frac{1}{2}\mathrm{Tr}G_{\mu\nu}G^{\mu\nu} - \frac{1}{2}\mathrm{Tr}V_{\mu\nu}V^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \\ &+ i\bar{L}_n\mathcal{D}^\mu\gamma_\mu L_n + i\bar{E}_n\mathcal{D}^\mu\gamma_\mu E_n + i\bar{Q}_n\mathcal{D}^\mu\gamma_\mu Q_n + i\bar{U}_n\mathcal{D}^\mu\gamma_\mu U_n + i\bar{D}_n\mathcal{D}^\mu\gamma_\mu D_n \\ &- (Y^l_{mn}\bar{L}_m H E_n + Y^d_{mn}\bar{Q}_m H D_n + Y^u_{mn}\bar{Q}_m\tilde{H}U_n + h.c.)\end{aligned}$$

$$+\mathcal{D}_\mu H^\dagger\mathcal{D}^\mu H - \lambda\left(H^\dagger H - \frac{v^2}{2}\right)^2$$

(all before the electroweak symmetry breaking all particles are massless!)

$$B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu,$$

$$V_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu - ig[V_\mu, V_\nu],$$

$$G_{\mu\nu} \equiv \partial_\mu G_\nu - \partial_\nu G_\mu - ig_s[G_\mu, G_\nu],$$

$$\mathrm{Tr}G_{\mu\nu}G^{\mu\nu} = \frac{1}{2}G^a_{\mu\nu}G^{a\,\mu\nu}, \quad \mathrm{Tr}V_{\mu\nu}V^{\mu\nu} = \frac{1}{2}V^i_{\mu\nu}V^{i\,\mu\nu}$$

Non-Abelian gluon/gauge boson stress tensors

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc}G^b_\mu G^c_\nu, \quad V^i_{\mu\nu} = \partial_\mu V^i_\nu - \partial_\nu V^i_\mu + g\epsilon^{ijk}V^j_\mu V^k_\nu$$

The Higgs boson doublet acquires a vacuum expectation values and breaks the electroweak symmetry ***spontaneously*** *down to U(1) electromagnetic (only photon and gluons are massless, other particles become massive!)*

$$H(x) = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} + \frac{h(x)}{\sqrt{2}} \end{pmatrix}$$

Gauge boson mass terms

$$\mathcal{D}_\mu H^\dagger\mathcal{D}^\mu H \longrightarrow \frac{g^2v^2}{4}W^+_\mu W^{\mu-} + \frac{(g^2+g'^2)v^2}{8}Z_\mu Z^\mu$$

$$M_W = \frac{gv}{2}, \quad M_Z = \frac{v\sqrt{g^2+g'^2}}{2} = \frac{M_W}{\cos\theta_W}$$

Fermion mass terms – due to Yukawa interactions

$$m_f = \frac{y_f}{\sqrt{2}}v$$

Final SM Lagrangian after the EW symmetry breaking

$$\mathcal{L}_{SM} = \mathcal{L}_{QCD} + \mathcal{L}^{free}_{lept} + \mathcal{L}_{f,em} + \mathcal{L}_{f,weak} + \mathcal{L}_Y + \mathcal{L}_V + \mathcal{L}_H + \mathcal{L}^{int}_{HV}$$

Strong interactions (QCD)

$$\mathcal{L}_{QCD} = -\frac{1}{4}G^a_{\mu\nu}G^{a\ \mu\nu} + \sum_{\text{quarks}} \bar{q}\left(i\gamma^\mu\partial_\mu - m_q - ig_s\frac{\lambda^a}{2}G^a_\mu\right)q$$

Free (kinetic) lepton part

$$\mathcal{L}^{free}_{lept} = \sum_n \bar{l}_n(i\gamma^\mu\partial_\mu - m_{l_n})l_n + \sum_n \bar{\nu}_n i\gamma^\mu\partial_\mu\nu_n$$

Electromagnetic interactions (QED)

$$\mathcal{L}_{f,em} = e\sum_f q_f\bar{f}\gamma^\mu A_\mu f$$

$$e = g\sin\theta_W = \frac{gg'}{\sqrt{g^2+g'^2}}$$

Weak interactions

$$\begin{aligned}\mathcal{L}_{f,weak} = &\frac{g}{2\sqrt{2}}\sum_n(\bar{\nu}_n\gamma^\mu(1-\gamma^5)W^+_\mu l_n + h.c.)\\ &+\frac{g}{2\sqrt{2}}\sum_{m,n}(\bar{u}_m\gamma^\mu(1-\gamma^5)W^+_\mu V_{mn}d_n + h.c.)\\ &+\frac{g}{2\cos\theta_W}\sum_f\bar{f}\gamma^\mu(t^f_3(1-\gamma^5) - 2q_f\sin^2\theta_W)fZ_\mu\end{aligned}$$

Higgs-fermion (Yukawa) terms

$$\mathcal{L}_Y = -\sum_f\frac{y_f}{\sqrt{2}}\bar{f}fh = -\sum_f\frac{m_f}{v}\bar{f}fh$$

Vector boson kinetic terms and interactions

$$\begin{aligned}\mathcal{L}_V = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \frac{M_Z^2}{2}Z_\mu Z^\mu\\ &-\frac{1}{2}|W^-_{\mu\nu}|^2 + M_W^2|W^-_\mu|^2 + \frac{g^2}{4}(W^-_\mu W^+_\nu - W^+_\mu W^-_\nu)^2\\ &-\frac{ig}{2}(F^{\mu\nu}\sin\theta_W + Z^{\mu\nu}\cos\theta_W)(W^-_\mu W^+_\nu - W^+_\mu W^-_\nu),\end{aligned}$$

$$W^-_{\mu\nu} \equiv (\partial_\mu + ieA_\mu + ig\cos\theta_W Z_\mu)W^-_\nu - (\mu\leftrightarrow\nu)$$

Higgs kinetic terms and scalar self-interactions

$$\mathcal{L}_H = \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}m_h^2 h^2 - \lambda v h^3 - \frac{\lambda}{4}h^4$$
$$m_h = \sqrt{2\lambda}v$$

Higgs-vector boson interactions

$$\mathcal{L}_{HV}^{int} = \frac{g^2}{2}vh|W_\mu^-|^2 + \frac{g^2+g^{'2}}{4}vhZ_\mu Z^\mu + \frac{g^2}{4}h^2|W_\mu^-|^2 + \frac{g^2+g^{'2}}{8}h^2 Z_\mu Z^\mu$$

Baryon number conservation – global Abelian U(1)b

$$q \to \mathrm{e}^{i\beta/3}q, \quad \bar{q} \to \mathrm{e}^{-i\beta/3}\bar{q}$$

$$(\nu_e,\ e) \to \mathrm{e}^{i\beta_e}(\nu_e, e), \quad (\bar{\nu}_e,\ \bar{e}) \to \mathrm{e}^{-i\beta_e}(\bar{\nu}_e,\ \bar{e})$$
$$(\nu_\mu,\ \mu) \to \mathrm{e}^{i\beta_\mu}(\nu_\mu, \mu), \quad (\bar{\nu}_\mu, \bar{\mu}) \to \mathrm{e}^{-i\beta_\mu}(\bar{\nu}_\mu, \bar{\mu})$$
$$(\nu_\tau,\ \tau) \to \mathrm{e}^{i\beta_\tau}(\nu_\tau, \tau), \quad (\bar{\nu}_\tau, \bar{\tau}) \to \mathrm{e}^{-i\beta_\tau}(\bar{\nu}_\tau, \bar{\tau})$$

Baryon number is conserved – but how to explain the baryon asymmetry in the Universe?

$$B = \frac{1}{3}(N_q - N_{\bar{q}})$$

Three conserved lepton numbers – but violated by neutrino oscillations!

$$L_e = (N_e + N_{\nu_e}) - (N_{e^+} + N_{\bar{\nu}_e}),$$
$$L_\mu = (N_\mu + N_{\nu_\mu}) - (N_{\mu^+} + N_{\bar{\nu}_\mu}),$$
$$L_\tau = (N_\tau + N_{\nu_\tau}) - (N_{\tau^+} + N_{\bar{\nu}_\tau}),$$