Ultra-high energy neutrinos in the Universe

If there exist neutrinos of ultra-high energies in Nature, they may scatter off relic neutrinos. The neutrino-neutrino cross section in the Standard Model of particle physics is very small. Its maximum value, $\sigma = 1.5 \cdot 10^{-31} \text{cm}^2$, is reached at center-of-mass energy $\sqrt{s} \approx M_Z \approx 90 \text{ GeV}$ when neutrinos pair-annihilate through the resonant Z-boson production. The observation of photons produced in Z-decay chains would be indirect evidence for the existence of relic neutrinos. Find the mean free path of ultra-high energy neutrino in the present Universe with respect to the above process. Does one expect a cutoff in the spectrum of ultra-high energy neutrinos, similar to the GZK cutoff in the spectrum of ultra-high energy protons (see Problem 1.8)?

Problem A5
useful GR math

Check the following properties of the Christoffel symbols and covariant derivative:

$$\Gamma^\mu_{\nu\mu} = \partial_\nu \ln \sqrt{-g},$$
$$g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = -\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\lambda\mu}),$$
$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} A^\mu),$$
$$\nabla_\mu A^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} A^{\mu\nu}), \quad \text{for} \quad A^{\mu\nu} = -A^{\nu\mu},$$
$$\nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi),$$

Problem No 2.6
Klein-Gordon equation for a scalar field in expanding Universe
Show that for a slowly varying scale factor, solutions to equation

\[
\frac{1}{a^2} \partial_\eta (a^2 \partial_\eta \phi) - \partial_i \partial_i + m^2 a^2 \phi = 0
\]

are superpositions of

\[
\phi(\eta, \vec{x}) = \frac{1}{a(\eta) \sqrt{\Omega(\eta)}} e^{i \int \Omega(\eta) d\eta} e^{-\vec{k} \cdot \vec{x}} \cdot (1 + \mathcal{O}(\partial_\eta a)),
\]

where \(\Omega(\eta) = \sqrt{k^2 + m^2 a^2(\eta)}\). Thus, coordinate frequency (derivative of the exponent with respect to conformal time \(\eta\)) equal \(\Omega(\eta)\), while the physical frequency is

\[
\omega(\eta) = \frac{\Omega(\eta)}{a(\eta)}.
\]