Lecture 2

Elements of General Relativity


1. Object of study in GR - curved 4-dim space-time interval

\[ ds^2 = g_{\mu\nu}(x)\, dx^\mu \, dx^\nu \quad \mu, \nu = 0, \ldots, 3 \]

\[ \Rightarrow 10 \text{ indep. functions} \]

\[ g_{\mu\nu}(x), \quad \mu \neq \nu \]

Minkowski signature \((+, - , - , - )\) \( \Rightarrow g = \text{det} (g_{\mu\nu}) < 0 \)

2. Basic principle of GR - all local coordinate frames are physically equivalent to each other.

Lorentz (coordinate) transformations, Lorentz group of coordinate transformations, \( x'^\mu \rightarrow x'^{\mu}(x) \) keep the interval unchanged \( ds = ds' \).

3. Consider objects (fields)

- scalar field \( \phi(x') = \phi(x) \) (scalar w.r.t. Lorentz transformations)

- contravariant vector \( A^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} \cdot A_\nu(x) \) (transforms as \( dx / dx' \))

- covariant vector \( A_\nu(x') = \frac{\partial x'^\nu}{\partial x^\mu} \cdot A^\mu(x) \) (transforms as \( \partial / \partial x' \))

Combination \( A^\mu B_\mu \rightarrow \) transforms as scalar!

- tensor of arbitrary rank, e.g.

3rd rank \( B^{\mu\nu\lambda}(x) = \frac{\partial x'^\mu}{\partial x^\rho} \cdot \frac{\partial x'^\nu}{\partial x^\sigma} \cdot \frac{\partial x'^\lambda}{\partial x^\tau} \cdot B^{\rho\sigma\tau}(x) \) (transforms as product of contravariant vectors)

Interval is invariant!

\[ g_{\mu\nu}(x)\, dx^\mu \, dx^\nu = g_{\mu\nu}(x)\, dx'\mu \, dx'\nu = g'_{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\rho} \cdot \frac{\partial x'^\nu}{\partial x^\sigma} \cdot g_{\rho\sigma}(x) \]

Contravariant \( g^{\mu\nu} \) tensor:

\[ g^{\mu\nu} = \delta^\mu_\nu \quad \text{diag} (1, 1, 1, 1) \]

\[ \uparrow \quad \text{Kronecker symbol} \]

Inverse to \( g_{\mu\nu} \) ! \( \Rightarrow \) operations of lowering/raising indices:

\[ A^\mu = g^{\mu\nu} A_\nu \]

\[ B_{\mu\nu} = g_{\mu\nu} B^{\rho\sigma} \]
In terms of Jacobian of transformations, \( J^a_b = \frac{\partial x^a}{\partial x^b} \), 

\[
\frac{\partial g}{\partial x^a} = J^b_a \frac{\partial g}{\partial x^b} \Rightarrow \text{Volume element in Minkowski space:} \quad V(x) = J^{-1}(x) \sqrt{-g} \quad \Rightarrow \quad J = \det(J^a_b)
\]

\[
\Rightarrow \quad V(x) = \sqrt{-g} \quad \text{d}^{n-1}x
\]

Levi-Civita symbol in Minkowski space, \( \epsilon^{a_1 \ldots a_n} \), is totally antisymmetric.

\[
A^\mu \epsilon^\mu_{\nu \rho} \quad \Rightarrow \quad \text{natural generalization:}
\]

\[
E^\mu_{\nu \rho} = \frac{1}{\sqrt{-g}} A^\mu \epsilon^\mu_{\nu \rho}
\]

Limit to Minkowski (flat) space-time: \( g_{\mu \nu} \Rightarrow g_{\mu \nu} = \text{diag}(-1,1,1,1) \).

Covariant derivative \( \nabla_\mu \rightarrow \) converts tensors into tensors.

\[
\nabla_\mu A^\mu = \partial_\mu A^\mu + \Gamma^\mu_{\nu \rho} A^\rho
\]

Consider parallel transport of the vector.

\[
A^\mu(x) = A^\mu(\tilde{x})
\]

\[
\tilde{A}^\mu(x) = A^\mu(x) - \left( \frac{\partial A^\mu}{\partial x^\lambda} \tilde{A}^\lambda(x) \right) dx^\lambda
\]

\[
\tilde{A}^\mu(x) = \frac{\partial A^\mu}{\partial x^\lambda} \tilde{A}^\lambda(x) + \frac{\partial^2 A^\mu}{\partial x^\lambda \partial x^\theta} \tilde{A}^\lambda(x) \ dx^\theta
\]

\[
\tilde{A}^\mu(x) = \tilde{A}^\mu(x) + \left( \frac{\partial A^\mu}{\partial x^\lambda} \tilde{A}^\lambda(x) \right) dx^\lambda
\]

Comparing with the original expression:

\[
\Gamma^\mu_{\nu \rho}(x) \equiv \frac{\partial A^\mu}{\partial x^\nu} \tilde{A}^\nu(x) - \frac{\partial A^\mu}{\partial x^\rho} \tilde{A}^\rho(x) + \sum_{\lambda=1}^4 \left( \frac{\partial A^\mu}{\partial x^\lambda} \frac{\partial A^\lambda}{\partial x^\nu} \tilde{A}^\nu(x) - \frac{\partial A^\mu}{\partial x^\lambda} \frac{\partial A^\lambda}{\partial x^\rho} \tilde{A}^\rho(x) \right)
\]

We define the covariant derivative as \( \nabla_\mu A^\mu(x) = \partial_\mu A^\mu(x) + \Gamma^\mu_{\nu \rho}(x) A^\rho(x) \).
3. Riemannian Geometry conditions (basic for GR1)

\[ \nabla^a (A^b) = \left( \nabla^a (\nabla^b A^c) \right) \rightarrow \nabla^a g_{ab} = 0 \]

\( \Gamma^a \) are metric connections \( \Rightarrow \) \( \Gamma^a g_{ab} = \Gamma^b g_{ap} + \Gamma^c g_{bp} \)

I symmetry \( \Rightarrow \) disappearance of the torsion tensor

\[ C^a_{\mu \nu} = \Gamma^a_{\mu \nu} - \Gamma^a_{\nu \mu} = 0 \]

and II define the Riemannian manifold and such \( \Gamma^a \) are Christoffel symbols

\[ \Gamma^a_{\mu \nu} = \frac{1}{2} g^{a \rho} \left( \partial_\mu g_{\rho \nu} + \partial_\nu g_{\rho \mu} - \partial_\rho g_{\mu \nu} \right) \]

Basic GR relations (to derive at home)

\[ \mathcal{L} A^\mu = \frac{1}{2} g^{\mu \rho} \left( \nabla^a (\nabla_\rho A^a) \right) \]

\[ \Delta^2 A^\mu = \frac{1}{2} g^{\mu \rho} \left( \nabla^a (\nabla_\rho A^a) \right) \]

Gauss law

\[ \int \left( \dfrac{1}{2} \, g_{\mu \nu} \, \nabla^\mu \xi^\nu \right) \, dx = \int \left( \xi^\nu \, g_{\mu \nu} A^\mu \right) \, dx = \int \left( \nabla^\mu A^\mu \right) \, dx \]

(Allows to perform integration by parts, element of the surface found)

6. Riemann tensor \( \Rightarrow \) measure of the curvature of the space-time

\( R^{\mu \nu \rho \sigma} = 0 \) in one frame, \( \neq 0 \) in another, so we have to define a quantity which characterize geometry itself (not the choice of the frame)

\[ [\nabla^\mu, \nabla^\nu] A^\lambda = \nabla^\mu \nabla^\nu A^\lambda - \nabla^\nu \nabla^\mu A^\lambda = A^\lambda R^{\mu \nu \rho \sigma} \]

[\text{Riemann tensor}]

\[ R^{\mu \lambda \nu \rho} = \frac{\partial A^\lambda}{\partial x^\rho} - \partial_\rho A^\lambda + \Gamma^\lambda_{\mu \sigma} \Gamma^\sigma_{\rho \nu} - \Gamma^\lambda_{\nu \sigma} \Gamma^\sigma_{\rho \mu} \]

\[ A^{a(12)} \neq A^{a(34)} \] in curved space

\[ R^{\mu \lambda \nu \rho} = \text{Ricci tensor} \]

\[ R = g^{\mu \nu} R_{\mu \nu} \] scalar curvature

\text{second order}

\[ \mathcal{L} \phi = R^{\mu \lambda \nu \rho} \]
1. Einstein equations

\[ g_{\mu\nu}(x) \leftrightarrow \text{gravitational field} \]

must be covariant, i.e., their form should not depend on the choice of the frame.

\[ \text{action functional} \quad S_S = 0 \quad \Rightarrow \quad \text{E. o. m. GR} \]

principle of extremum

\[ \varphi_{\mu\nu} \rightarrow -\Lambda = \text{const} \quad \Rightarrow \quad S_{\Lambda} = -\Lambda \int d^4x \sqrt{-g} \]

simplest possibility

\[ S_{\Lambda} = -\Lambda \int d^4x \sqrt{-g} \]

\[ S_{\Lambda} = (\text{mass})^2 \]

\[ S_{\Lambda} \] is not enough!

**Einstein - Hilbert action:**

\[ S_{EH} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R \]

depends on the second derivatives of \( g_{\mu\nu} \)

Total action of the gravitational field:

\[ S_{Sg} = S_{\Lambda} + S_{EH} \]

Employing

\[ \det (M + \Lambda M) = \det (M) \frac{1}{1} + \frac{1}{2} \text{tr} (M^{-1} \Lambda M) + h.c. \]

we obtain in \( S_{Sg} \): linear term:

\[ \delta g = g_{\mu\nu} \delta g_{\mu\nu} \quad \Rightarrow \quad S_{\Lambda} = -\Lambda \int d^4x \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} \]

\[ S_{EH} = S_1 + S_2 + S_3 \]

2. \( S_1 = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R \left( \frac{1}{g} \right) = -\frac{1}{32\pi G} \int d^4x \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} \]

3. \( S_2 = 0 \rightarrow g_{\mu\nu} \delta g_{\mu\nu} = g_{\mu\nu} \delta g^{\mu\nu} \rightarrow S_{\mu\nu} = -g_{\mu\nu} \delta g^{\mu\nu} \)

4. \( S_3 = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \nabla_{\mu} \left( S \Gamma_{\mu\nu}^{\lambda} \right) - \nabla_{\nu} \left( S \Gamma_{\mu\nu}^{\lambda} \right) \)

III. \( \delta \nabla_{\mu} \rightarrow \text{tensor!} \quad S_{R_{\mu\nu}} = \nabla_{\lambda} \left( S \Gamma_{\mu\nu}^{\lambda} \right) - \nabla_{\nu} \left( S \Gamma_{\mu\nu}^{\lambda} \right) \)

using \( \checkmark \)

\[ \delta S_3 = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \delta R_{\mu\nu} \]

\[ \Rightarrow \quad \delta S_{R_{\mu\nu}} \text{ transforms into integral over surface} \rightarrow 0 \quad \text{(ask to Gauss law)} \]

\[ \delta S > \text{does not contribute to E. o. m.} \]
Therefore,
\[
SS_{\text{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \right) \delta g_{\mu \nu}
\]
Einstein tensor $G_{\mu \nu}$

\[
R_{\mu \nu} = \frac{1}{2} g_{\mu \nu} R = 8\pi G \Lambda g_{\mu \nu}
\]

Einstein equations in vacuum:

In the presence of the matter fields serving as sources of the gravitational field, the matter

\[
S_{m} = \int d^4x \sqrt{-g} \mathcal{L}_m \quad \mathcal{L}_m = \mathcal{L}_m (\phi, \partial \phi)
\]

interact with gravity:

\[
S_{m} = \frac{1}{2} \int d^4x \sqrt{-g} \left( T_{\mu \nu} \right) \delta g_{\mu \nu}
\]

(energy-momentum tensor)

\[
(R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) = 8\pi G \left( \Lambda g_{\mu \nu} + T_{\mu \nu} \right)
\]

Example 1: scalar field theory

arbitrary function

\[
\mathcal{L}_\phi = \frac{1}{2} g_{\mu \nu} \partial \phi \partial \phi - V(\phi)
\]

\[
\Rightarrow \quad [T_{\mu \nu}^\phi = \partial\phi \partial \phi \eta_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \delta_{\phi} \eta_{\phi}]
\]

Example 2: electromagnetic field coupled to gravity

\[
\mathcal{L}_m = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} \sqrt{-g} \quad F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\]

field strength tensor

\[
[\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu]
\]

\[
\Rightarrow \quad T_{\mu \nu}^{\text{em}} = -F_{\mu \alpha} F^{\alpha \nu} \eta_{\mu \nu} + \frac{1}{2} g_{\mu \nu} F^{\alpha \beta} F_{\alpha \beta}
\]

We notice that $T_{\mu \nu}^{\text{em}}$ and $T_{\mu \nu}^\phi$ coincide with corresponding tensors in Minkowski space (up to a divergence) of an antisymmetric tensor:

Conservation law in curved space-time:

\[
\nabla_\mu T_{\mu \nu}^{\text{em}} = 0
\]

\[
\nabla_\mu T_{\mu \nu}^\phi = 0
\]

and one could check that $\nabla_\mu (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) = 0$

Remark! in flat space $\partial_\mu T_{\mu \nu}^{\text{em}} = 0$ implies existence of four conserved quantities

$\nabla_\mu T_{\mu \nu}^{\text{em}} = 0$ does not imply the existence of the integrals of motion, which could be interpreted as $\{E, \mathbf{P}\}$ of the system!

Concept of energy-momentum is NOT defined in GR!