Lecture 5. Friedmann equation. Sample cosmological solutions.

Consider expanding Universe filled with matter. We treat matter as ideal fluid (isotropic, w/o internal rotation). In the rest frame of matter, the energy-momentum tensor is diagonalized (due to isotropy).

\[ T^\mu_\nu = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & -\rho \end{pmatrix} \]

In flat space, \( u^\mu = (1, 0, 0, 0) \) in the rest frame.

\[ \Rightarrow \text{we define } T^\mu_\nu = (p + \rho) u^\mu u_\nu - \rho g^\mu_\nu \quad \text{as energy-momentum tensor of matter} \]

In arbitrary frame, \( g^\mu_\nu \rightarrow \text{gauge} \).

with covariant conservation

\[ \nabla_\mu T^\mu_\nu = 0 \]

\[ T_{00} = (p + \rho) u_0 u_0 - \rho_0 \rho \]

\[ u^0 = 1, \quad u_0 = 1 \]

\[ \Rightarrow T_{00} = p \]

In comoving frame:

\[ T_{00} = (p + \rho) u_0 u_0 - \rho_0 \rho \]

\[ u^0 = 1, \quad u_0 = 1 \]

\[ \Rightarrow T_{00} = p \]

\[ R_{00} - \frac{2}{3} R \approx 0 \]

\[ \Rightarrow \frac{\dot{a}}{a} = \text{constant} \]

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\[ \Rightarrow R_{ij} = \left( \frac{\dot{a}}{a} + 2 \frac{a}{a} \dot{a} \right) \delta_{ij} \]

\[ R = \frac{\dot{a}}{a^2} R_{00} = R_{00} - \frac{1}{2} \delta_{ij} R_{ij}, \quad \Rightarrow \delta_{ij} R_{ij} = 3 \]

\[ \Rightarrow R = -6 \left( \frac{\dot{a}}{a} + \frac{a}{a} \frac{\dot{a}}{a} + \frac{\dot{a}}{a} \right) \]

\[ \Rightarrow \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{\dot{a}}{a} \]

Friedmann equation relates expansion rate \( \ddot{a} \) with total energy density \( \rho \) and spatial curvature.

\[ \ddot{a} - \frac{8\pi G}{3} \rho - \frac{\dot{a}}{a} = 0 \]

\[ \Rightarrow \] two unknown functions, \( a(t) \) and \( \rho(t) \)

\[ \Rightarrow \text{additional equations are needed!} \]

Covariant conservation of \( T^\mu_\nu \):

\[ \nabla_\mu T^\mu_\nu = 0 \]
\[ 3 \text{rd equation} \rightarrow \text{the equation of state of matter} \]

\[ P = \rho \left( \frac{8 \pi G}{3} \rho \right) \]

not a consequence of GR.

Eqs. 1, 2, 3 completely determine dynamics of the cosmological expansion.

\[ p = P(\rho) \]

\[ P = 0, P = P/3, P = -P \]

\[ \rho = \sum \rho_i \]

potentials:

\[ \frac{dP}{dt} = -3 d(lu) \]

\[ s = \text{const} \]

\[ \text{entropy conservation in comoving volume} \]

One can show that other Einstein equations are identically satisfied for solutions of (1) and (2) and later we will see that entropy density with vanishing chemical potential.

Cosmological solutions

To a good approx. consider spatially flat Universe \( \mathbb{R} \times S^3 \).

Friedmann equation:

\[ \left( \frac{a'}{a} \right)^2 = \frac{8 \pi G}{3} \rho \]

\[ \rho = \text{const} \]

Solutions include two constants: \( a_0 \) and \( t_0 \).

1. Non-relativistic matter or "dust"

\[ p = 0 \rightarrow \text{Eq. (2)} \rightarrow \rho = \text{const} \]

\[ a(t) = \text{const} \left( t - t_0 \right)^{2/3} \]

\[ \rho(t) = \frac{\text{const}}{(t-t_0)^2} \]

The moment of Big Bang \((t=t_0)\) arbitrary: const.

\[ a < 0 ! \]
Of course, we cannot extrapolate the classical solution to \( t \to \infty \) at \( \rho_0 \approx M_p^2 \times 10^{96} \text{ GeV}^4 \). Classical gravity laws are not applicable.

We count time from the cosmological singularity \( t = 0 \) — the age of the Universe.

\[
{\rho} = \frac{3}{8\pi G} \frac{a^2}{H^2} = \frac{1}{6\pi G} \frac{1}{t^2}
\]

Hubble parameter

\[
H(t) = \frac{a'_0}{a(t)} = \frac{2}{3t}
\]

for non-relativistic matter domination, the age of the Universe contradicts the bound to \( > 1.3 \times 10^{10} \) years?

\( \Rightarrow \) The Universe has not always been dominated by dust!

The cosmological horizon — region of the Universe causally connected by the time \( t \).

\( \Rightarrow \) An observer living at time \( t \) cannot know, in principle, what has happened outside the light-like geodesics obeying \( ds^2 = 0 \) — the size of observable part of the Universe.

\[
\begin{align*}
\rho(t) & = a(t) \gamma(t) = a(t) \int_0^t \frac{dt'}{a(t')} = 3t = \frac{2}{H(t)} \Rightarrow \text{now} \quad H_0 = \frac{2}{H_0} \\
\rho(t) & = M_P^2 \times 10^{96} \text{ GeV}^4
\end{align*}
\]

\( \Rightarrow \) Relativistic matter or radiation.

\[
\begin{align*}
\frac{a}{a'} & = \text{const} \quad \Rightarrow \quad \rho = \frac{\text{const}}{a^4} \quad (\text{due to dilution of the number density of the energy of each particle before \( \Delta \approx 1/\ell \))}
\end{align*}
\]

\( H(t) = \frac{t}{2t} \Rightarrow \quad \rho = \frac{3}{8\pi G} \frac{1}{H^2(t)} = \frac{3}{32\pi G} \frac{1}{t^2}
\]

\( \ell_{\text{h}}(t) = a(t) \left( \frac{dt}{a(t)} \right) = 2t = \frac{1}{H(t)} \)
Assuming thermal equilibrium and neglecting chemical potential:

\[ P = \frac{T^4}{3\pi^2} \rho \]

\[ \rho_0 = \frac{\pi}{6} \rho_0 T^d + \frac{2}{3} \rho_0 T^d \]

effective number of degrees of freedom d.o.f.

for \( M_{\text{pl}} \leq T \)

\[ H = \frac{T^2}{M_{\text{pl}}^2} \]

relation of the expansion rate with temperature

\[ M_{\text{pl}} = \sqrt{\frac{G}{8\pi}} \approx 1.66 \times 10^{19} \text{ GeV} \]

Comparing (1) and (2), we see:

\[ \frac{T(t)}{\alpha(t)} \approx \text{const} \]

\[ \frac{T}{T} \approx -H = -\frac{T^2}{M_{\text{pl}}^2} \]

cooling rate

behaves in the same way as for the relativistic non-interacting matter with Planckian spectrum (see previous lecture)

The vacuum equation of state has an exotic form:

\[ P = -\rho \]

or for not very high curvature:

\[ T_{ij} = -p \delta_{ij} \]

The vacuum equation of state has a positive energy density and negative pressure.

One of the biggest challenges in fundamental physics is to calculate \( \rho \) from the Particle Physics.

\[ T_{\mu\nu} = \rho \text{ in any frame} \]

The conservation law \( \nabla_{\mu} T^{\mu\nu} = 0 \) is consistent with \( \rho = \text{const} \) for \( \rho = \text{const} \).

Also, in the absence of matter

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \Lambda g_{\mu\nu} \]

we can identify \( \Lambda = \rho \text{ const} \).

\[ \frac{d^2 a}{dt^2} = -\frac{H^2}{3} + \frac{\Lambda}{3} a \]

\[ H = \sqrt{\frac{8\pi}{3}} \rho_{\text{mass}} = \text{const} \]

\[ \text{de Sitter space} \Rightarrow \ddot{a} > 0 \]
de Sitter space does not have the initial singularity $t \to -\infty$, $a \to 0$, and the metric can be non-singular! "the beginning of time"

the cosmological (particle) horizon is absent:

$$\ell_H(t) = a(t) \left( \int_{-\infty}^{t} \frac{dt'}{a(t')} \right) = \infty$$

For de-Sitter-like spaces, "event horizon" notion is used instead!

$$\ell_{\text{dS}} = a(t) \left( \int_{t}^{\infty} \frac{dt'}{a(t')} \right) = \frac{1}{H_{\text{dS}}}$$

distance from which signals at the moment $t' = t$ will ever reach the observer at $x = 0$ and $t = t'$

time-like geodesics $\dot{x} = 0$ $\Rightarrow$ The conformal size of the physical size at time $t$

region is $\ell_{\text{dS}} = a(t) \left( \int_{0}^{\infty} \frac{dt'}{a(t')} \right)$

de Sitter (event) horizon

**IV** Barotropic e.o.s. $p = \omega \rho$, $\omega > -1$

models with $\Omega_c \ll 1$ attract considerable attention in recent years (quintessence, time-dependent $\Lambda$-term, etc.)

\begin{align*}
\text{covariant conservation} & \quad \Rightarrow \quad \rho = \frac{\text{const}}{a^3(1+\omega)} \\
\text{Friedmann eqn.} & \quad \Rightarrow \quad a = \text{const.} \left( \frac{t}{a} \right)^{\alpha} \\
\text{there is cosmological singularity at } t = 0 \quad \Rightarrow \quad \alpha = \frac{2}{3} \left( \frac{1}{1+\omega} \right) \\
\dot{a} = \text{const} \cdot \alpha (\alpha-1) \dot{a}^{\alpha-2} & \quad \Rightarrow \quad \dot{a} < 0 \quad \text{for } \alpha < 1 \\
& \quad \Rightarrow \quad \dot{a} > 0 \quad \text{for } \alpha > 1
\end{align*}

\begin{align*}
\Omega_c > -\frac{1}{3} & \quad \text{deceleration (I)} \\
\Omega_c < -\frac{1}{3} & \quad \text{acceleration (II)}
\end{align*}

for open Universe ($\alpha = -1$) $\Rightarrow \rho = \frac{\text{const}}{a^2}$

$\Rightarrow \omega = -\frac{1}{3}$, $\dot{a} = 0$

\begin{enumerate}
\item have particle horizon, and do not have event horizon ($\alpha < 1$)
\item have event horizon, and do not have particle horizon
\end{enumerate}
Solutions with recollapse → expansion of the Universe is followed by contraction.

Example: The closed cosmological model with non-relativistic matter:

\[
\left(\frac{\ddot{a}}{a}\right)^2 = \frac{8\pi G}{3} \frac{\rho}{a^2} - \frac{1}{a^2} = 0 \quad \text{when} \quad \rho = \rho_m \quad \text{The maximal scale factor of the Universe given by the total mass of matter}
\]

\[
\dot{a} = a^{-\frac{1}{2}} \left( \frac{\rho_m \sin^2 \frac{\theta}{2}}{a^2} \right) \quad \text{singularity} \quad \rightarrow \quad \theta = 2\pi \quad \text{it goes back to singularity!}
\]

In physical time,

\[
t = \int a(\theta) \, d\theta = \frac{\rho_m}{2} \left( \theta - \sin \theta \right) \rightarrow \left( \theta = \frac{\pi}{2} \right)
\]

Similar situation happens when $\Lambda$-term becomes negative.