Nature of Dark Energy - one of the major problems of contemporary natural science.

Among numerous hypotheses \( \Rightarrow \) cosmological constant = vacuum energy density \( \rho = \omega \rho \) with \( \omega = \frac{1}{3} \)

is considered to be the most common one due to the existence of "quintessence" - spatially homogeneous field

Dynamical Dark Energy \( \Rightarrow \) time dependence of \( \omega = \omega(t) \)

I. Evolution of scalar field in expanding Universe

\[ S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2} \left( \nabla \phi \right)^2 - V(\phi) \right] \]

We consider spatially flat Universe with real scalar field in curved space-time

\[ S_{0} = - \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \tilde{\phi}(p) \tilde{\phi}(-p) = - \frac{\partial V}{\partial \phi} \]

for spatially homogeneous field \( \phi = \phi(t) \) \( \Rightarrow \) [\( \frac{\dot{\phi}}{\phi} + 3H\phi = -\frac{\partial V}{\partial \phi} \)]

"Fast roll" regime \( \dot{H}\phi \ll \phi \) - friction is weak

"particle" rapidly rolls down

"slow roll" regime (friction is strong!) \( \dot{H}\phi \ll \phi \) - change of the field \( \dot{\phi} \sim \dot{\phi}H^{-1} \) \( \sqrt{\phi\dot{\phi}H^2} \ll \phi \) is small Hubble time

Power-law potentials:

\[ V(\phi) = \frac{1}{2} m^2 \phi^2, \quad V(\phi) = \frac{1}{2} \phi^4 \]

\[ \sqrt{\phi\dot{\phi}H^2} \ll \phi \]

second "slow-roll" condition:

\[ \ddot{\phi} \ll \dot{\phi} \rightarrow \ddot{\phi} \sim \dot{\phi}H \frac{\ddot{\phi}}{H} \]

\[ \frac{\ddot{\phi}}{H} \ll \dot{\phi} \quad \frac{\dot{\phi}}{H} \ll \phi \]

\[ \ddot{\phi} \ll \dot{\phi} \quad \frac{\dot{\phi}}{H} \ll \phi \]

field remains practically constant during the evolution \( \ddot{\phi} \ll \dot{\phi} \)

see problem 4.7
Consider how \( \phi(t) \) approaches minimum of the potential

\[
V(\phi) = \frac{m^2}{2} \phi^2 \quad \Rightarrow \quad \ddot{\phi} + \frac{2}{a^2} \frac{\dot{a}}{a} \dot{\phi} + m^2 \phi = 0
\]

Let's get rid of the friction term by replacement:

\[
\phi(t) = \frac{1}{a^3} \chi(t) \quad \Rightarrow \quad \ddot{\chi} + \left( \frac{2}{m^2} \frac{\dot{a}}{a} - \frac{3}{4} \frac{\dot{a}^2}{a^2} \right) \chi = 0
\]

Note:

\[
\frac{\dot{a}}{a} \sim \frac{\dot{a}^2}{a^2} = H^2 \quad \text{if} \quad m^2 \ll H^2 \quad \Rightarrow \quad \text{the} \, \text{"slow-rollof" \ regime!}
\]

In the opposite case,

\[
m^2 \gg H^2 \quad \Rightarrow \quad \chi = 0
\]

\[
\phi(t) = \phi_0 \cdot \frac{\cos(mt + p)}{a^3} \quad \phi \text{ oscillates \ oscillation equation near minimum}
\]

Consider action

\[
S = \int d\chi \quad a^3 \left( \frac{1}{2} \dot{\chi}^2 - \frac{1}{2} \frac{\dot{a}^2}{a^2} (\phi_0 \chi)^2 - \frac{m^2}{2} \phi_0^2 \right)
\]

If \( \partial_\chi \chi = 0 \) and \( a(t) = \text{const} \) \( \Rightarrow \) \( E = \int d\chi \quad a^3 \left( \frac{1}{2} \dot{\chi}^2 + \frac{m^2}{2} \phi_0^2 \right) \)

For slow (adiabatic) change of \( a(t) \), energy is conserved.

\[
\Rightarrow \quad \text{solution still oscillates with amplitude } \propto a^{3/2}
\]

Since

\[
a^3(t) \left( \frac{1}{2} \dot{\phi}^2 + \frac{m^2}{2} \phi^2 \right) = \text{const}
\]

Generally, energy is not conserved in time-dependent \( a(t) \) backgrounds.

Energy - momentum tensor:

\[
\frac{\partial \epsilon}{\partial x} = \frac{\partial \rho}{\partial x} = \frac{\partial p}{\partial x} = 0
\]

For \( \phi = \phi(t) \), in the locally-Lorentz frame

\[
\phi^\mu = (\phi^0, \phi^1, \phi^2, \phi^3)
\]

In the slow-roll regime

\[
\frac{\dot{\phi}^2}{(1/2) \ddot{\phi}^2} \ll (1/2) \ddot{\phi}^2 - V(\phi) \ll \dot{\phi}^2 \ll \frac{H^2}{m^2}
\]

This means

\[
P_{\phi} = -p_{\phi} \approx V(\phi)
\]

in the fast-roll regime (oscillations) \( \Rightarrow \)

\[
\rho \gg H^2
\]

in the vacuum e.o.s. approximately (more precisely \( \rho > p \))
\[
T_{\alpha\beta} = \frac{m^2 \phi^2}{2} \frac{1}{a^3(t)} = \frac{m^2 \phi^2}{2} \frac{1}{a^3(t)} \cos(2\omega t + 2\phi) \delta_{ij}
\]

Averaging over the period oscillations yields:

\[
T_{\alpha\beta} = \frac{m^2 \phi^2}{2} \frac{1}{a^3(t)} \delta_{ij} \Rightarrow T_{\phi\phi} = \frac{m^2 \phi^2}{2} \frac{1}{a^3(t)} = 0 \Rightarrow P_\phi = 0 \Rightarrow P_\phi \propto a^{-3}(t)
\]

The oscillating scalar field \(\phi\) can be interpreted as a collection of free particles of mass \(m\) with number density \(n = \frac{P_\phi}{m} \propto a^{-3}(t)\).

Accelerated expansion due to scalar field:

- May be explained by scalar field in the slow roll regime.
- Homogeneous, \(\phi = \phi(t)\), natural in inflationary theories.

Let's find conditions for the accelerated expansion:

- Assume \(V(\phi) \propto \phi^2\) is not very large.

If energy density of the scalar field dominates in the Universe:

\[
\text{Friedmann equation} \quad H^2 = \frac{8\pi}{3M_P^2} \rho_\phi = \frac{8\pi}{3M_P^2} V(\phi)
\]

However,

\[
V(\phi) \approx \rho_\phi (\approx 0.73 \rho_\text{cr}) \quad \text{very small!}
\]

In our Universe:

\[
\frac{\dot{\phi}^2}{\phi^2} \ll H^2 \quad \Rightarrow \quad \phi \gg M_P \quad \Rightarrow \quad \text{slow roll condition for power-law potentials}
\]

The power-law potential must be extremely flat to provide the accelerated expansion,

\[
V(\phi) = \lambda \phi^4 \quad \Rightarrow \quad \lambda \lesssim 10^{122}
\]

For example, \(V(\phi) = \frac{m^2 \phi^2}{2} \frac{1}{a^3(t)} \Rightarrow m \lesssim \frac{\sqrt{\rho_\phi}}{M_P} \sim 10^{-33} \text{eV}\)

Quintessence, thus, works for very exotic scalar potentials only!

- Zeroth vacuum energy \(\rho\) is the biggest puzzle of time-dependent quintessence; \(\rho \propto \Phi\) applies...

Dark Energy
What mechanism ensures the "correct" present value of the scalar field?

"Tracker" models

$V(q) = \frac{M^{n+4}}{nq^n}$

for $a(t) \propto t^\alpha$; $\dot{q} + \frac{3\alpha}{t} q - \frac{M^{n+4}}{q^{n+1}} = 0$

$N = \text{small}$ \hspace{1cm} (n > 2)

$\Rightarrow$ "tracker" solution:

$\dot{q} = C(n, \alpha) M^{(n+2)/2}$

Power-law attractor$^*$ with $\frac{1}{n+2}$

$s_0 = \dot{q}_0 > \dot{q}^{tr}(t_i)$

solution $\dot{q}(t)$ rolls faster and approaches $\dot{q}^{tr}(t)$ again.

$\Rightarrow$ in sufficiently large range of initial data, any solution $q(t)$ approaches the "tracker" solution $\dot{q}^{tr}(t)$.

1. $\dot{q} \propto V(q) \propto \frac{1}{t^{2-2\alpha}}$ \hspace{1cm} $\Rightarrow$ not the slow-roll regime!

2. $\frac{\dot{s}}{s^{tot}} \propto t^{2\alpha}$ \hspace{1cm} $\Rightarrow$ increases with time!

3. $a \propto t^\alpha \Rightarrow \dot{q}^{tr} \propto \frac{1}{a^{2-2\alpha}}$, then $\dot{w} = \frac{\dot{q}}{q} \cdot \frac{\dot{q}}{q}$ with

$w = -1 + \frac{\frac{1}{3} - \frac{2}{n+2}}{\frac{1-2\alpha}{n+2}}$

$\Rightarrow$ w is the parameter of the e.o.s. of the dominant matter ($w = \frac{1}{3}$ for relativistic, $w = 0$ for non-relativistic matter)

$\Rightarrow$ e.o.s. of the "tracker" field depends on the e.o.s. of the dominant matter.

Solution $\circ$ is valid at times when $\dot{q} \ll \dot{p}_m$, grows with time.

at later times $\dot{q} \gg \dot{p}_m$ $\Rightarrow$ slow roll regime $\Rightarrow$ accelerated expansion.

Solution $\circ$ is no longer valid.
Scalar field starts to dominate over matter when \( V(\phi) \sim \phi^2 \sim \frac{\rho}{\rho_m} \).

\[ \Rightarrow \rho \sim \frac{\phi^2}{\rho_m} \]

Matter density \( \rho_m \) is:

\[ \rho_m = \frac{\frac{3}{8\pi G}}{M_{Pl}^2 H(t)} \sim \frac{M_{Pl}^2}{\rho_m^2} \]

So for \( \rho \sim \rho_m \):

\[ \phi \sim M_{Pl} \] (while initial (arbitrary) value \( \phi_i \ll M_{Pl} \)).

Existing data is consistent with both: cosmological constant and with "tracker" field and with many other quintessence models.