Lecture 9: Thermodynamics of expanding Universe

Thermal equilibrium: if rates of interactions are much faster than Universe expansion rate, 
\[ \downarrow \] freezing out (going off thermal equilibrium) \[ \uparrow \] time of freezing out? 
\[ \text{direction of inequilibrium processes?} \]

We need to know laws of equilibrium thermodynamics (TD)

TD description \[ \to \] in terms of chemical potentials

\[ A_1 + A_2 + \ldots + A_n \leftrightarrow B_1 + B_2 + \ldots + B_n \] (potential energy)

\[ \text{species} \Rightarrow \begin{cases} \mu_{A_1} + \mu_{A_2} + \ldots + \mu_{A_n} = \mu_{B_1} + \mu_{B_2} + \ldots + \mu_{B_n} \end{cases} \]

chemical equilibrium

Examples:

\[ e^- + e^- \rightarrow e^- + e^- \rightarrow \text{elastic scattering} \]
\(L \rightarrow L = 0\) (new state is produced)

annihilation \[ \Rightarrow e^+ + e^- \rightarrow 2\gamma \]
\[ \mu_{e^+} = -\mu_{e^-} \]

\[ \mu_L = 0 \]

opposite chemical potentials for conserved set of quantum numbers \(Q^{(i)}\)

Consider conserved set of quantum numbers \(Q^{(i)}\)

\[ \mu_A = \sum_i \mu_i Q_A^{(i)} \]

\(Q^{(i)} = \{B, L_e, \mu^i, L^i, Q^j\}\)

e.g. for up-quark: \[ \mu_u = \frac{1}{3} \mu_B + \frac{2}{3} \mu_B \]

\(B_{\text{aryon}}\) and \(e_{\text{lectric}}\) number charge

In the momentum space (for relatively weak interactions) \(\Delta\)

\[ P^2 = \sum_i \left( P^2_{i} \right) \]

\[ E(P^2) = \sqrt{P^2 + m^2} \]

other Bose-gas (\(n\)) and Fermi-gas (\(n\))
For large $\exp$: Maxwell-Boltzmann distribution

$$f(p) = \frac{1}{(2\pi)^{3/2}} e^{-(E(p) - \mu)/T}$$

Rest energy works for low density gas of non-relativistic particles, with $m - p \gg T$.

$$f(p) = \frac{p}{(2\pi)^{3/2}} e^{-(p^2/2m)} e^{-p^2/2kT}$$

Kinetic energy term.

By definition, $n_i = \int f(p) d^3p$ is the number density of particles of type $i$.

$$dN = \frac{1}{4\pi} \int dE \sqrt{E^2 - m^2} E \frac{d^3p}{p}$$

Integrate over angles.

$$d^3p = p^2 \sin \theta dp \sin \theta d\theta dp$$

$$\theta^2 = \phi^2 + \phi^2$$

$$\theta = \phi^2$$

$$\phi = 0$$

$$\phi = 2$$

$$\int dp = 2\pi$$

$$\int d\theta = 2\pi$$

$$E dE = p dp$$

$$f(p) = \frac{2}{(2\pi)^{3/2}} e^{-(p^2/2m)} e^{-p^2/2kT}$$

$g_8 = g_{e^+} = 2, g_0 = g_\gamma = 1$. $w^+, w^0 \rightarrow 3$ pol. states.

In early Universe, chemical potentials are very small.

$E(p) \gg m \rightarrow$ net positive charge of quarks is compensated by the net positive charge of electrons (Universe is neutral).

$E^2 dE$ is energy density.

$\rho_i = \int f(p) E(p) d^3p = \frac{4\pi}{3} \int_0^\infty (p^2/2m) f(p) E(p) dp = \frac{4\pi}{3} \int_0^\infty (p^2/2m) f(p) dp$.

Pressure

$$P = \frac{\Delta p_{tot}}{\Delta s \Delta t}$$

Total transferred momentum.

$$\Delta p = \frac{p_{final} - p_{initial}}{\Delta s \Delta t}$$

Number of particles.

$$\Delta s = \frac{p_{final} - p_{initial}}{\Delta s \Delta t}$$

Pressure.

$$\rho_i = \int 2p_2 \cdot \frac{p_2}{E} f(p) d^3p = \frac{4\pi}{3} \int_0^\infty \left( \frac{p^2}{2m} \right) f(p) E(p) dp = \frac{4\pi}{3} \int_0^\infty f(p) (E^2 - m^2)^{3/2}$$

$$\Delta p_2 = 2p_2$$
Physically interesting cases:

\[ \text{gas of relativistic particles} \quad T \gg m_i, \quad p_i = 0 \quad \Rightarrow \quad p_i = \frac{q_i}{2\pi^2} \int \frac{E^3}{e^{E/T} - 1} \, dE = \sqrt{\frac{2}{\pi}} \frac{q_i}{30} T^4 \quad \Rightarrow \text{Bose-Einstein}\]

\[ \frac{7}{8} \frac{q_i}{30} T^4 \quad \Rightarrow \text{Fermi-Diraclaw} \]

For cosmological plasma in thermal equilibrium:

\[ n_i = \frac{q_i}{2\pi^2} \int \frac{E^2}{e^{E/T} - 1} \, dE = \frac{1}{8} \frac{S(3)}{\pi^2} T^3 \quad \Rightarrow \text{Bose-Einstein}\]

\[ \frac{7}{8} \frac{q_i}{2\pi^2} \int \frac{E^2}{e^{E/T} - 1} \, dE = \frac{1}{8} \frac{S(3)}{\pi^2} T^3 \quad \Rightarrow \text{Fermi-Dirac}\]

**Example:** Consider cosmic matter in thermal equilibrium with electromagnetic interactions.

- E.g. Compton scattering
  - \( e^+ e^- \) annihilation

By dimensional analysis:

\[ \Gamma \sim \alpha^2 T \]

Thermal equilibrium condition:

\[ \frac{\Gamma}{H(T)} \]

Expansion rate:

\[ H(T) = \frac{T^2}{M_{\text{Pl}}} \quad \Rightarrow \quad T \ll c^2 M_{\text{Pl}} \sim 10^{14} \text{ GeV} \]

\[ \text{plasma in thermal eq.} \]

\[ n_i = \frac{q_i}{2\pi^2} \left( \frac{m_i T}{2\pi} \right)^{3/2} e^{-\frac{m_i}{T}} \quad \Rightarrow \]

\[ p_i = \frac{q_i}{2} n_i T \quad \Rightarrow \text{e.o.s.} \quad P = \frac{\rho}{3} + O(T/m_i) \]
Entropy of expanding Universe → one of the main thermo-dynamical characteristics

The first law of TD:
(for variable number of particles)

Introducing densities

\[ p = \frac{E}{V} \quad n = \frac{N}{V} \quad s = \frac{S}{V} \quad \Rightarrow \]

for a \( V \)-constant region

\[ T_ds = dp - \mu \, dm \]

\[ \Rightarrow \quad \int dV \rightarrow \text{for the entire system} \]

\[ s = \frac{P + P - \mu N}{T} \]

(2) Relativistic matter \( \mu = 0 \)

\[ s = \frac{P + P}{T} \quad \Rightarrow \quad s_i = \frac{4}{3} \frac{P_i}{T} = \left\{ \begin{array}{l} \frac{\text{g} \, 2n_i \, n}{45} \quad T^3 \quad \Rightarrow \text{Bose} \\
\frac{\text{g} \, 2n_i \, n}{46} \quad T^3 \quad \Rightarrow \text{Fermi} \end{array} \right. \]

(b) Non-relativistic case:

\[ n_i = \rho_i \left( \frac{m_i}{2 \pi} \right)^{3/2} e^{\frac{m_i - \mu_i}{T}} \quad \Rightarrow \quad \mu_i = \frac{m_i}{\text{g}_i} \left( \frac{2\pi}{m_i T} \right)^{3/2} \]

\[ s_i = n_i \left\{ \frac{5}{2} + \log \left[ \frac{\text{g}_i \left( \frac{m_i}{2 \pi} \right)^{3/2}}{m_i T} \right] \right\} \quad \text{(proportional to} \ n_i ) \]

\( n_i \) for non-relativistic particles is very small

\( \text{e.g. at} \ T \lesssim 100 \text{MeV}, \) protons and neutrons are non-relativistic

\( N_B \sim 10^{-8} N \)

\( \text{at} \ T \lesssim 0.5 \text{MeV}, \) electrons are non-relativistic

\( \Rightarrow \) so their contribution is very small

Total entropy density:

\[ s = \left\{ \begin{array}{l} \frac{2n_i^2}{45} \quad T^3 \quad \text{(for anisotropic pressure)} \\
\frac{2n_i}{45} \quad T^3 \quad \text{(for isotropic pressure)} \end{array} \right. \]

Second law of TD:

Entropy of any closed system can only increase \( \mathcal{S} \) and it stays constant for equilibrium evolution.

Let's see how it works in an expanding universe...
coming back to: \( dE = Td^3B - p dV + \sum_i \mu_i dN_i \) \( \text{comoving volume} \quad V = \frac{a^3}{\rho} \)
due to conservation of quantum numbers in comoving volume \( dN_i = 0 \)
\[ \Rightarrow T \frac{dS}{dt} = T \frac{d(\frac{a^3S}{\rho})}{dt} = (p + 3p) dV + V dp = \]

entropy in comoving volume is preserved
\[ \Rightarrow \frac{a^3}{\rho} = \text{const.} \quad \text{(in equilibrium)} \]
where we used \( \frac{p}{\rho} + 3 \frac{a}{a_t} (p + 3p) = 0 \)

\( \Rightarrow \) can be used for study time-independent characteristics of asymmetries of conserved quantum numbers

\( e.g. \) at \( T \lesssim 100 \text{ GeV} \) \( \Rightarrow \) no baryon number violation so if it is conserved in comoving volume
\[ \frac{(n_B - n_{\bar{B}})}{a^3} = \text{const.} \quad \text{time-independent characteristics} \]

\[ \Delta B = \frac{n_B - n_{\bar{B}}}{s} \]

Let's estimate it!
baryon-to-photon ratio \( \eta_B = \text{const}(t) \)
\[ \eta_B \sim \Delta B \]
\[ \frac{S_0}{T_0^3} = \frac{2 \pi^2}{45} \left( \frac{g^*}{2} \right)^{\frac{3}{2}} \]
\[ T_0 \approx M_{\nu_e} \]
present value \( g_s^* = 2 + \frac{4}{3} g_s^* = \frac{43}{20} T_8 \)
\[ T_8 \approx 5 \times 10^9 \text{ K} \]

BBN/CMB studies: \( \eta_B = 6.2 \times 10^{-10} \)
\[ \Rightarrow \Delta B = \frac{n_B}{s} \eta_B = \frac{2 \times 5}{45} \times 10^{-10} \]

present density of photons \( n_g = 2 \times \frac{\epsilon(3)}{n^2} \times \frac{g_s^*}{T_0^3} = 411 \text{ cm}^{-3} \)
\[ n_B = n_B \eta_B \text{, } \eta_B > \bar{\eta}_B = \frac{m_B n_B}{P_c} \]
\[ P_c = 0.5 \times 10^{-5} \text{ GeV} \]

\( \Delta_B \approx 0.046 \) \( \Rightarrow \) baryon chemical potential is very small