Particle Physics Phenomenology
2. Phase space and matrix elements

Torbjörn Sjöstrand

Department of Astronomy and Theoretical Physics
Lund University
Sölvegatan 14A, SE-223 62 Lund, Sweden

NBI, Copenhagen, 3 October 2011
Four-vectors

four-vector: \( \mathbf{p} = (E; \mathbf{p}) = (E; p_x, p_y, p_z) \)

vector sum: \( \mathbf{p}_1 + \mathbf{p}_2 = (E_1 + E_2; \mathbf{p}_1 + \mathbf{p}_2) \)

vector product: \( \mathbf{p}_1 \mathbf{p}_2 = E_1 E_2 - \mathbf{p}_1 \mathbf{p}_2 \)
\[ = E_1 E_2 - p_{x1} p_{x2} - p_{y1} p_{y2} - p_{z1} p_{z2} \]
\[ = E_1 E_2 - |\mathbf{p}_1| |\mathbf{p}_2| \cos \theta_{12} \]

square: \( \mathbf{p}^2 = E^2 - \mathbf{p}^2 = E^2 - p_x^2 - p_y^2 - p_z^2 = m^2 \)

transverse mom.: \( p_\perp = \sqrt{p_x^2 + p_y^2} \)

transverse mass: \( m_\perp = \sqrt{m^2 + p_x^2 + p_y^2} = \sqrt{m^2 + p_\perp^2} \)
\[ E^2 = m^2 + \mathbf{p}^2 = m^2 + p_\perp^2 + p_z^2 = m_\perp^2 + p_z^2 \]

Warning: No standard to distinguish \( \mathbf{p} = (E; p_x, p_y, p_z) \) and \( \mathbf{p} = |\mathbf{p}| = \sqrt{p_x^2 + p_y^2 + p_z^2} \), but usually clear from context.
When we remember, we will try to use \( p = |\mathbf{p}| \), since \( \bar{p} = \mathbf{p} \).
Decay widths and cross sections

Decay width at rest, $1 \rightarrow n$:

$$d\Gamma = \frac{|M|^2}{2M} \ d\Phi_n$$

Integrated it gives exponential decay rate

$$\frac{dP}{dt} = \Gamma e^{-\Gamma t} \quad \text{and} \quad \langle \tau \rangle = 1/\Gamma$$

Collision process cross section, $2 \rightarrow n$:

$$d\sigma = \frac{|M|^2}{4\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}} \ d\Phi_n$$

Integrated it gives collision rate

$$N = \sigma \int \mathcal{L}(t) \ dt \quad \text{with} \quad \mathcal{L} \approx f \frac{n_1 n_2}{A}$$

in a theorist’s approximation of the luminosity $\mathcal{L}$ for a collider.
\[ d\Phi_n = (2\pi)^4 \delta^{(4)}(P - \sum_{i=1}^{n} p_i) \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i} \]

Lorentz covariant:

\[ d^4 p_i \delta(p_i^2 - m_i^2) \theta(E_i) = d^4 p_i \delta(E_i^2 - (p_i^2 + m_i^2)) \theta(E_i) = \frac{d^3 p_i}{2E_i} \]

with \( E_i = \sqrt{p_i^2 + m_i^2} \) and using

\[ \delta(f(x)) = \sum_{x_j, f(x_j) = 0} \frac{1}{|f'(x_j)|} \delta(x - x_j) \]

Application: Lorentz invariant production cross sections \( E \frac{d\sigma}{d^3 p} \)
Spherical symmetry

Spherical coordinates:

\[
\frac{d^3 p}{E} = \frac{dp_x}{p} \frac{dp_y}{p} \frac{dp_z}{p} = \frac{p^2 dp d\Omega}{E} = \frac{E dp dE d\Omega}{E} = p dE d\Omega
\]

where \( \Omega \) is the unit sphere,

\[
d\Omega = d(cos \theta) d\phi = sin \theta d\theta d\phi
\]

\[
p_x = p \ sin \theta \ cos \varphi
\]

\[
p_y = p \ sin \theta \ sin \varphi
\]

\[
p_z = p \ cos \theta
\]

and \( E^2 = p^2 + m^2 \Rightarrow E dE = p dp. \)

Convenient for use e.g. in resonance decays, but not for standard QCD physics in \( pp \) collisions. Instead:
Cylindrical coordinates:

\[
\frac{d^3 p}{E} = \frac{dp_x \, dp_y \, dp_z}{E} = \frac{d^2 p_\perp \, dp_z}{E} = d^2 p_\perp \, dy
\]

with rapidity \( y \) given by

\[
y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z} = \frac{1}{2} \ln \frac{(E + p_z)^2}{(E + p_z)(E - p_z)} = \frac{1}{2} \ln \frac{(E + p_z)^2}{m^2 + p_\perp^2}
\]

\[
= \ln \frac{E + p_z}{m_\perp} = \ln \frac{m_\perp}{E - p_z}
\]

The relation \( dy = dp_z/E \) can be shown by

\[
\frac{dy}{dp_z} = \frac{d}{dp_z} \left( \ln \frac{E + p_z}{m_\perp} \right) = \frac{d}{dp_z} \left( \ln \left( \sqrt{m_\perp^2 + p_z^2} + p_z \right) - \ln m_\perp \right)
\]

\[
= \frac{1}{2} \frac{2p_\perp}{\sqrt{m_\perp^2 + p_z^2}} + 1
\]

\[
= \frac{p_z + E}{E + p_z} = \frac{1}{E}
\]
Introduce (lightcone) $p^+ = E + p_z$ and $p^- = E - p_z$.

Note that $p^+ p^- = E^2 - p_z^2 = m_\perp^2$.

Consider boost along $z$ axis with velocity $\beta$, and $\gamma = 1/\sqrt{1 - \beta^2}$.

\[
\begin{align*}
\begin{aligned}
    p'_{x,y} &= p_{x,y} \\
    p'_z &= \gamma (p_z + \beta E) \\
    E' &= \gamma (E + \beta p_z) \\
    p'^+ &= \gamma (1 + \beta) p^+ = \sqrt{\frac{1 + \beta}{1 - \beta}} p^+ = k \ p^+ \\
    p'^- &= \gamma (1 - \beta) p^+ = \sqrt{\frac{1 - \beta}{1 + \beta}} p^- = \frac{p^-}{k} \\
    y' &= \frac{1}{2} \ln \frac{p'^+}{p'^-} = \frac{1}{2} \ln \frac{k \ p^+}{p'^- / k} = y + \ln k \\
    y'_2 - y'_1 &= (y_2 + \ln k) - (y_1 + \ln k) = y_2 - y_1
\end{aligned}
\end{align*}
\]
If experimentalists cannot measure $m$ they may assume $m = 0$. Instead of rapidity $y$ they then measure pseudorapidity $\eta$:

$$y = \frac{1}{2} \ln \frac{\sqrt{m^2 + p^2 + p_z}}{\sqrt{m^2 + p^2 - p_z}} \Rightarrow \eta = \frac{1}{2} \ln \frac{|p| + p_z}{|p| - p_z} = \ln \frac{|p| + p_z}{p_\perp}$$

or

$$\eta = \frac{1}{2} \ln \frac{p + p \cos \theta}{p - p \cos \theta} = \frac{1}{2} \ln \frac{1 + \cos \theta}{1 - \cos \theta} = \frac{1}{2} \ln \frac{2 \cos^2 \theta/2}{2 \sin^2 \theta/2} = \ln \frac{\cos \theta/2}{\sin \theta/2} = - \ln \tan \theta/2$$

which thus only depends on polar angle.

$\eta$ is **not** simple under boosts: $\eta'_2 - \eta'_1 \neq \eta_2 - \eta_1$.

You may even flip sign!

Assume $m = m_\pi$ for all charged $\Rightarrow y_\pi$; intermediate to $y$ and $\eta$. 
By analogy with \( \frac{dy}{dp_z} = \frac{1}{E} \) it follows that \( \frac{d\eta}{dp_z} = \frac{1}{p} \).

Thus

\[
\frac{d\eta}{dy} = \frac{d\eta}{dp_z} \frac{dy}{dp_z} = \frac{E}{p} > 1
\]

with limits

\[
\frac{d\eta}{dy} \to \frac{m_\perp}{p_\perp} \text{ for } p_z \to 0
\]
\[
\frac{d\eta}{dy} \to 1 \text{ for } p_z \to \pm\infty
\]

so if \( dn/dy \) is flat for \( y \approx 0 \) then \( dn/d\eta \) has a dip there.

\[
\eta - y = \ln \frac{p + p_z}{p_\perp} - \ln \frac{E + p_z}{m_\perp} = \ln \frac{p + p_z}{E + p_z} \frac{m_\perp}{p_\perp} \to \ln \frac{m_\perp}{p_\perp} \text{ when } p_z \gg m_\perp
\]
Evaluate in rest frame, i.e. $P = (E_{cm}, 0)$.

\[
d\Phi_2 = (2\pi)^4 \delta^{(4)}(P - p_1 - p_2) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2}
\]

\[
= \frac{1}{16\pi^2} \delta(E_{cm} - E_1 - E_2) \frac{d^3 p_1}{E_1 E_2}
\]

\[
= \frac{1}{16\pi^2} \delta(\sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2} - E_{cm}) \frac{p^2 dp d\Omega}{E_1 E_2}
\]

\[
= \frac{1}{16\pi^2} \frac{p}{E_1 + E_2} \frac{p^2 dp d\Omega}{E_1 E_2}
\]

\[
= \frac{E_1 E_2}{16\pi^2} \frac{p d\Omega}{E_1 + E_2 E_1 E_2}
\]

\[
= \frac{p d\Omega}{16\pi^2 E_{cm}}
\]
The Källén function – 1

\[ \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2} = E_{cm} \]

gives solution

\[ E_1 = \frac{E_{cm}^2 + m_1^2 - m_2^2}{2E_{cm}} \]
\[ E_2 = \frac{E_{cm}^2 + m_2^2 - m_1^2}{2E_{cm}} \]
\[ p = \frac{1}{2E_{cm}} \sqrt{(E_{cm}^2 - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2} = \frac{1}{2E_{cm}} \sqrt{\lambda(E_{cm}^2, m_1^2, m_2^2)} \]

where Källén \( \lambda \) function is

\[ \lambda(a^2, b^2, c^2) = (a^2 - b^2 - c^2)^2 - 4b^2c^2 \]
\[ = a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 \]
\[ = (a^2 - (b + c)^2)(a^2 - (b - c)^2) \]
\[ = (a + b + c)(a - b - c)(a - b + c)(a + b - c) \]
Hides everywhere in kinematics, e.g.

\[
d\sigma = \frac{|\mathcal{M}|^2}{4\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}} \, d\Phi_n
\]

has

\[
4((p_1 p_2)^2 - m_1^2 m_2^2) = (p_1^2 + 2p_1 p_2 + p_2^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2
\]
\[
= ((p_1 + p_2)^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2
\]
\[
= \lambda(E_{\text{cm}}^2, m_1^2, m_2^2)
\]

so

\[
d\sigma = \frac{|\mathcal{M}|^2}{2\sqrt{\lambda(E_{\text{cm}}^2, m_1^2, m_2^2)}} \, d\Phi_n
\]
Mandelstam variables

For process $1 + 2 \rightarrow 3 + 4$

\[
\begin{align*}
  s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\
  t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\
  u &= (p_1 - p_4)^2 = (p_2 - p_3)^2
\end{align*}
\]

In rest frame, massless limit: $m_1 = m_2 = m_3 = m_4 = 0$,

\[
\begin{align*}
  p_{1,2} &= \frac{E_{\text{cm}}}{2} (1; 0, 0, \pm) \\
  p_{3,4} &= \frac{E_{\text{cm}}}{2} (1; \pm \sin \hat{\theta}, 0, \pm \cos \hat{\theta}) \\
  s &= E_{\text{cm}}^2 \\
  t &= -2p_1p_3 = -\frac{s}{2} (1 - \cos \hat{\theta}) \\
  u &= -2p_2p_4 = -\frac{s}{2} (1 + \cos \hat{\theta}) \quad s + t + u = 0
\end{align*}
\]
Mandelstam variables with masses

\[ \beta_{34} = \frac{\sqrt{\lambda(s, m_3^2, m_4^2)}}{s} \]

\[ p_{3,4} = \frac{\sqrt{s}}{2} \left( 1 \pm \frac{m_3^2 - m_4^2}{s} ; \pm \beta_{34} \sin \hat{\theta}, 0, \pm \beta_{34} \cos \hat{\theta} \right) \]

\[ t = m_1^2 + m_3^2 - \frac{s}{2} \left( 1 + \frac{m_1^2 - m_2^2}{s} \right) \left( 1 + \frac{m_3^2 - m_4^2}{s} \right) + \frac{s}{2} \beta_{12} \beta_{34} \cos \hat{\theta} \]

\[ d\sigma = \frac{|M|^2}{2 \sqrt{\lambda(s, m_1^2, m_2^2)} \sqrt{s}} \frac{p_{34}}{16\pi^2} \frac{d\cos \hat{\theta} d\varphi}{2s \beta_{12}} = \frac{|M|^2}{8\pi} \frac{\beta_{34}}{2s \beta_{12}} \frac{d\cos \hat{\theta}}{8\pi} \]

assuming no polarization \( \Rightarrow \) no \( \varphi \) dependence

\[ \frac{d\sigma}{dt} = \frac{d\sigma}{d\cos \hat{\theta}} \frac{d\cos \hat{\theta}}{dt} = \frac{|M|^2}{16\pi s^2 \beta_{12}^2} \]
Mandelstam variables with final-state masses

Usually $m_{1,2} \approx 0$, while often $m_{3,4}$ non-negligible

$$t, u = -\frac{1}{2} \left[ s - m_3^2 - m_4^2 \mp s \beta_{34} \cos \hat{\theta} \right]$$

$$\frac{d\sigma}{dt} = \frac{|M|^2}{16\pi s^2}$$

$$s + t + u = m_3^2 + m_4^2$$

$$tu = \frac{1}{4} \left[ (s - m_3^2 - m_4^2)^2 - s^2 \beta_{34}^2 \cos^2 \hat{\theta} \right]$$

$$= \frac{1}{4} \left[ s^2 \beta_{34}^2 + 4m_3^2m_4^2 - s^2 \beta_{34}^2 \cos^2 \hat{\theta} \right]$$

$$= \frac{1}{4} s^2 \beta_{34}^2 \sin^2 \hat{\theta} + m_3^2m_4^2 = sp_{\perp}^2 + m_3^2m_4^2$$

$$p_{\perp}^2 = \frac{tu - m_3^2m_4^2}{s}$$
Classify $2 \to 2$ diagrams by character of propagator, e.g.

$$d\sigma(qq' \to qq') \sim \frac{8\pi\alpha_s^2}{9t^2} = \frac{32\pi\alpha_s^2}{9s^2(1 - \cos\theta)^2} = \frac{8\pi\alpha_s^2}{9s^2 \sin^4\hat{\theta}/2} \approx \frac{8\pi\alpha_s^2}{9p_\perp^2}$$

i.e. Rutherford scattering!

Singularities reflect channel character, e.g. pure $t$-channel:

$$\frac{d\sigma(qq' \to qq')}{dt} = \frac{\pi}{s^2} \frac{4\alpha_s^2 s^2 + u^2}{9 t^2}$$

peaked at $t \to 0 \Rightarrow u \approx -s$, so
Order-of-magnitude cross sections

With masses neglected:

\[ s\text{-channel} : \quad \frac{d\sigma}{dt} \sim \frac{\pi}{s^2} \]

\[ t\text{-channel, spin 1} : \quad \frac{d\sigma}{dt} \sim \frac{\pi}{t^2} \]

\[ t\text{-channel, spin } \frac{1}{2} : \quad \frac{d\sigma}{dt} \sim \frac{\pi}{-st} \]

\[ u\text{-channel} : \quad \text{same with } t \to u \]

Add couplings at vertices:

\[ \text{qqg} : \quad C_F \alpha_s \]

\[ \text{ggg} : \quad N_c \alpha_s \]

\[ \text{ff}\gamma : \quad e_f^2 \alpha_{\text{em}} \]

\[ \text{ff}'W : \quad |V_{ff'}|^2 \frac{\alpha_{\text{em}}}{4 \sin^2 \theta_W} \]

\[ \text{ff}'Z : \quad (v_f^2 + a_f^2) \frac{\alpha_{\text{em}}}{16 \sin^2 \theta_W \cos^2 \theta_W} \]
Consider \( q(1) g(2) \rightarrow q(3) g(4) \):

\[
|\mathcal{M}|^2 = t : p_{g^*} = p_1 - p_3 \Rightarrow m_{g^*}^2 = (p_1 - p_3)^2 = t \Rightarrow \frac{d\sigma}{dt} \sim \frac{1}{t^2}
\]

\[
u : p_{q^*} = p_1 - p_4 \Rightarrow m_{q^*}^2 = (p_1 - p_4)^2 = \nu \Rightarrow \frac{d\sigma}{dt} \sim -\frac{1}{\nu u}
\]

\[
s : p_{q^*} = p_1 + p_2 \Rightarrow m_{q^*}^2 = (p_1 + p_2)^2 = s \Rightarrow \frac{d\sigma}{dt} \sim \frac{1}{s^2}
\]

Contribution of each sub-graph is gauge-dependent, only sum is well-defined:

\[
\frac{d\sigma}{dt} = \frac{\pi \alpha_s^2}{s^2} \left[ \frac{s^2 + u^2}{t^2} + \frac{4}{9} \frac{s}{(-u)} + \frac{4}{9} \frac{(-u)}{s} \right]
\]
What $Q^2$ scale to use for $\alpha_s = \alpha_s(Q^2)$?

Should be characteristic virtuality scale of process!

But e.g. for $qg \to qg$: both $s$-, $t$- and $u$-channel + interference.

At small $t$ the $t$-channel graph dominates $\Rightarrow Q^2 \sim |t|$, 

at small $u$ the $u$-channel graph dominates $\Rightarrow Q^2 \sim |u|$, 

in between all graphs comparably important $\Rightarrow Q^2 \sim s \sim |t| \sim |u|$. 

Suitable interpolation:

\[
Q^2 = p_\perp^2 = \frac{tu}{s} \quad \rightarrow \quad -t \text{ for } t \to 0
\]

\[
\quad \rightarrow \quad -u \text{ for } u \to 0
\]

\[
\quad \rightarrow \quad \frac{s}{4} \text{ for } t = u = -\frac{s}{2}
\]

but could equally well be multiple of $p_\perp^2$, or more complicated $\Rightarrow$ one limitation of LO calculations.
Resonance shape given by Breit-Wigner

\[ 1 \leftrightarrow \rho(s) = \frac{1}{\pi} \frac{m\Gamma}{(s - m^2)^2 + m^2\Gamma^2} \]

\[ \mapsto \frac{1}{\pi} \frac{s\Gamma(m)/m}{(s - m^2)^2 + s^2\Gamma^2(m)/m^2} \]

where \( m \mapsto \sqrt{s} \) in phase space and \( \Gamma(s) \mapsto \Gamma(m) \sqrt{s}/m \) for gauge bosons, neglecting thresholds.

Latter shape suppressed below and enhanced above peak; tilted. For \( s \to 0 \) \( \rho(s) \) goes to constant or like \( s \).

PDF’s tend to be peaked at small \( x \): convolution enhances small \( s \).

Can give secondary mass-spectrum “peak” in \( s \to 0 \) region.

But note that

\[ |\mathcal{M}|^2 = |\mathcal{M}_{\text{signal}} + \mathcal{M}_{\text{background}}|^2 \]

so in many cases Breit-Wigner cannot be trusted except in the neighbourhood of the peak, where signal should dominate.
Three-body phase space

Three-body final states has $3 \cdot 3 - 4$ degrees of freedom. In massless case straightforward to show that, in CM frame,

$$d\Phi_3 = (2\pi)^4 \delta^{(4)}(P - p_1 - p_2 - p_3) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} \frac{d^3 p_3}{(2\pi)^3 2E_3}$$

$$= \frac{1}{8(2\pi)^5} dE_1 dE_2 d\cos \theta_1 d\varphi_1 d\varphi_{21}$$

with $\theta_1, \varphi_1$ polar coordinates of 1 and $\varphi_{21}$ azimuthal angle of 2 around 1 axis (Euler angles). Phase space limits $0 \leq E_{1,2} \leq E_{\text{cm}}/2$ and $E_1 + E_2 = E_{\text{cm}} - E_3 > E_{\text{cm}}/2$.

Same simple phase space expression holds in massive case, but phase space limits much more complicated!

Higher multiplicities increasingly difficult to understand. One solution: recursion!
Factorized three-body phase space

Drop factors of $2\pi$, and don’t write implicit integral signs. Introduce intermediate “particle” $12 = 1 + 2$.

$$d\Phi_3(P; p_1, p_2, p_3) \sim \delta^{(4)}(P - p_1 - p_2 - p_3) \frac{d^3p_1}{2E_1} \frac{d^3p_2}{2E_2} \frac{d^3p_3}{2E_3} \delta^{(4)}(p_{12} - p_1 - p_2) \, d^4p_{12}$$

$$= \delta^{(4)}(P - p_{12} - p_3) \, d^4p_{12} \frac{d^3p_3}{2E_3} \left[ \delta^{(4)}(p_{12} - p_1 - p_2) \frac{d^3p_1}{2E_1} \frac{d^3p_2}{2E_2} \right]$$

$$= \delta^{(4)}(P - p_{12} - p_3) \, d^4p_{12} \delta(p_{12}^2 - m_{12}^2) \, dm_{12}^2 \frac{d^3p_3}{2E_3} \, d\Phi_2(p_{12}; p_1, p_2)$$

$$= \, dm_{12}^2 \left[ \delta^{(4)}(P - p_{12} - p_3) \frac{d^3p_{12}}{2E_{12}} \frac{d^3p_3}{2E_3} \right] \, d\Phi_2(p_{12}; p_1, p_2)$$

$$= \, dm_{12}^2 \, d\Phi_2(P; p_{12}, p_3) \, d\Phi_2(p_{12}; p_1, p_2)$$

Note: here 4 angles + 1 mass$^2$; last slide 3 angles + 2 energies.
Recursive phase space

Generalizes to

\[ d\Phi_n(P; p_1, \ldots, p_n) = d\Phi_2(P; p_1, p_2) \times d\Phi_{n-1}(P; p_1, \ldots, p_{n-1}) \]

Can be viewed as a sequential decay chain, with undetermined intermediate masses.

Recall \( d\Phi_2(P; p_1, p_2) \propto \sqrt{\lambda(M^2, m_1^2, m_2^2)} \)

where \( d\Omega_{12} \) is the unit sphere in the 1+2 rest frame.

Now can write down e.g. 4-body phase space:
$$d\Phi_4(P; p_1, p_2, p_3, p_4) \propto \frac{\sqrt{\lambda(M^2; m_4^2, m_{123}^2)}}{M^2} m_{123} \, dm_{123} \, d\Omega_{1234} \times \frac{\sqrt{\lambda(m_{123}^2; m_3^2, m_{12}^2)}}{m_{123}^2} m_{12} \, dm_{12} \, d\Omega_{123} \frac{\sqrt{\lambda(m_{12}^2; m_1^2, m_2^2)}}{m_{12}^2} \, d\Omega_{12}$$

Mass limits coupled, but can be decoupled: pick two random numbers $0 < R_{1,2} < 1$ and order them $R_1 < R_2$. Then

$$\Delta = M - (m_1 + m_2 + m_3 + m_4)$$

$$m_{12} = m_1 + m_2 + R_1 \Delta$$

$$m_{123} = m_1 + m_2 + m_3 + R_2 \Delta$$

uniformly covers $dm_{12} \, dm_{123}$ space with weight

$$\frac{\sqrt{\lambda(M^2; m_4^2, m_{123}^2)}}{M} \frac{\sqrt{\lambda(m_{123}^2; m_3^2, m_{12}^2)}}{m_{123}} \frac{\sqrt{\lambda(m_{12}^2; m_1^2, m_2^2)}}{m_{12}}$$
For massless case a smart solution is RAMBO (RAndom Momenta and BOosts), which is 100% efficient:

1. Pick $n$ massless 4-vectors $p_i$ according to

$$E_i e^{-E_i} d\Omega_i$$

2. boost all of them by a common boost vector that brings them to their overall rest frame

3. rescale them by a common factor that brings them to the desired mass $M$

Can be modified for massive cases, but then no longer 100% efficiency; gets worse the bigger $\sum m_i / M$ is.

MAMBO: workaround for high multiplicities
Efficiency troubles

Even if you can pick phase space points uniformly, $|\mathcal{M}|^2$ is not! A $n$-body process receives contributions from a large number of Feynman graphs, plus interferences. Can lead to extremely low Monte Carlo efficiency.

Intermediate resonances $\Rightarrow$ narrow spikes when $(p_i + p_j)^2 \approx M_{\text{res}}^2$.

$t$-channel graphs $\Rightarrow$ peaked at small $p_{\perp}$.

Multichannel techniques:

$$|\mathcal{M}|^2 = \frac{\sum_i |\mathcal{M}_i|^2}{\sum_i |\mathcal{M}_i|^2} \sum_i |\mathcal{M}_i|^2$$

so pick optimized for either $|\mathcal{M}_i|^2$ according to their relative integral, and use ratio as weight.

Still major challenge in real life!
In reality all beams are composite:

\[ p : q, g, \bar{q}, \ldots \]
\[ e^- : e^-, \gamma, e^+, \ldots \]
\[ \gamma : e^\pm, q, \bar{q}, g \]

\[ \sigma^{AB} = \sum_{i,j} \int \int d x_1 \ d x_2 \ f_i^{(A)}(x_1, Q^2) \ f_j^{(B)}(x_2, Q^2) \hat{\sigma}_{ij} \]

\( x \): momentum fraction, e.g. \( p_i = x_1 p_A; p_j = x_2 p_B \)
\( Q^2 \): factorization scale, “typical momentum transfer scale”

Factorization only proven for a few cases, like \( \gamma^*/Z^0 \) production, and strictly speaking not correct e.g. for jet production, but good first approximation and unsurpassed physics insight.
Subprocess kinematics

If \( p_A + p_B = (E_{cm}; 0) \), \( A, B \) along \( \pm z \) axis, and 1, 2 collinear with \( A, B \) then conveniently put them massless:

\[
p_1 = (E_{cm}/2)(1; 0, 0, 1)
\]
\[
p_2 = (E_{cm}/2)(1; 0, 0, -1)
\]

such that \( \hat{s} = (p_1 + p_2)^2 = x_1 x_2 s = \tau s \). Velocity of subsystem is

\[
\beta_z = \frac{p_z}{E} = \frac{x_1 - x_2}{x_1 + x_2}
\]

and its rapidity

\[
y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z} = \frac{1}{2} \ln \frac{x_1}{x_2}
\]

d\( x_1 \) d\( x_2 = d\tau \) dy convenient for Monte Carlo.

Historically \( x_F = 2p_z/E_{cm} = x_1 - x_2 \).

Subprocess 2 \( \rightarrow \) 2 kinematics for \( \hat{\sigma} \): \( \hat{s}, \hat{t}, \hat{u} \).
\[ L \Rightarrow \text{Feynman rules} \Rightarrow \text{Matrix Elements} \Rightarrow \text{Cross Sections} \]

\[ + \text{Kinematics} \Rightarrow \text{Processes} \Rightarrow \ldots \Rightarrow \]

(Higgs simulation in CMS)
QCD at Fixed Order

Distribution of observable: $O$

In production of $X +$ anything

\[
\left. \frac{d\sigma}{dO} \right|_{\text{ME}} = \sum_{k=0} \int d\Phi_{X+k} \left| \sum_{\ell=0} M_{X+k}^{(\ell)} \right|^2 \delta(O - O(p)_{X+k})
\]

Fixed Order (all orders)

- Cross Section differentially in $O$
- Phase Space
- Sum over "anything" $\approx$ legs
- Matrix Elements for $X$+$k$ at ($\ell$) loops
- Momentum configuration
- Sum over identical amplitudes, then square

Truncate at $k=n$, $l=0$

$\rightarrow$ Leading Order for $X + n$

Lowest order at which $X + n$ happens
Loops and Legs

Another representation

<table>
<thead>
<tr>
<th>Loops</th>
<th>Legs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X^{(2)}$</td>
<td>$X+1^{(2)}$</td>
</tr>
<tr>
<td>$X^{(1)}$</td>
<td>$X+1^{(1)}$</td>
</tr>
<tr>
<td>Born</td>
<td>$X+1^{(0)}$</td>
</tr>
</tbody>
</table>
Loops and Legs

Another representation

Loops

$X^{(2)}$  $X+1^{(2)}$  ...

$X^{(1)}$  $X+1^{(1)}$  $X+2^{(1)}$  $X+3^{(1)}$  ...

Born  $X+1^{(0)}$  $X+2^{(0)}$  $X+3^{(0)}$  ...

Legs

Note: $\sigma \rightarrow \infty$ if both jets not resolved
Loops and Legs

Another representation

<table>
<thead>
<tr>
<th>Loops</th>
<th>Legs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X^{(1)}$</td>
<td>$X+1^{(0)}$</td>
</tr>
<tr>
<td>$X^{(1)}$</td>
<td>$X+2^{(0)}$</td>
</tr>
<tr>
<td>$X^{(2)}$</td>
<td>$X+1^{(2)}$</td>
</tr>
<tr>
<td>$X^{(2)}$</td>
<td>$X+1^{(2)}$</td>
</tr>
</tbody>
</table>

Note: $X+1$ jet observables only correct at LO

X @ NLO
(includes $X+1$ @ LO)
Loops and Legs

Another representation

Loops

$X^{(2)}$ $X+1^{(2)}$ ...

$X^{(1)}$ $X+1^{(1)}$ $X+2^{(1)}$ $X+3^{(1)}$ ...

Born $X+1^{(0)}$ $X+2^{(0)}$ $X+3^{(0)}$ ...

Lags

$X+1@NLO$
(includes $X+2@LO$)

Note: $\sigma \rightarrow \infty$
if no jet resolved

Note: $X+2$ jet observables
only correct at $LO$
Loops and Legs

Another representation

\[ X^{(2)} \quad X+1^{(2)} \quad \ldots \]

\[ X^{(1)} \quad X+1^{(1)} \quad X+2^{(1)} \quad X+3^{(1)} \quad \ldots \]

\[ \text{Born} \quad X+1^{(0)} \quad X+2^{(0)} \quad X+3^{(0)} \quad \ldots \]

**X @ NNLO**
(includes \(X+1 @ NLO\))
(includes \(X+2 @ LO\))

\[ \sigma \rightarrow \sigma_{\text{NNLO}} \]
if no jet resolved

Note: \(X+2\) jet observables only correct at LO
Stating the problem(s)

- Multi-particle final states for signals & backgrounds.
- Need to evaluate $d\sigma_N$:

$$\int_{\text{cuts}} \left[ \prod_{i=1}^{N} \frac{d^3q_i}{(2\pi)^3 2E_i} \right] \delta^4 \left( p_1 + p_2 - \sum_i q_i \right) |M_{p_1p_2\rightarrow N}|^2.$$

- Problem 1: Factorial growth of number of amplitudes.
- Problem 2: Complicated phase-space structure.
- Solutions: Numerical methods.
Example for factorial growth: $e^+ e^- \rightarrow q\bar{q} + ng$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$#\text{diags}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
</tr>
<tr>
<td>4</td>
<td>384</td>
</tr>
</tbody>
</table>

Remember: to be squared for number of squared MEs.
Basic ideas of efficient ME calculation

Need to evaluate $|M|^2 = \left| \sum_i M_i \right|^2$

- Obvious: Traditional textbook methods (squaring, completeness relations, traces) fail
  - ⇒ result in proliferation of terms ($M_i M_j^*$)
  - ⇒ Better: Amplitudes are complex numbers,
  - ⇒ add them before squaring!

- Remember: spinors, gamma matrices have explicit form could be evaluated numerically (brute force)
  - But: Rough method, lack of elegance, CPU-expensive
Helicity method

- Introduce basic helicity spinors (needs to "gauge" vectors)
- Write everything as spinor products, e.g.
  \[ \bar{u}(p_1, h_1)u(p_2, h_2) = \text{complex numbers} \]
  Completeness rel'n: \( (p + m) \implies \frac{1}{2} \sum_h \left[ \left( 1 + \frac{m^2}{p^2} \right) \bar{u}(p, h)u(p, h) + \left( 1 - \frac{m^2}{p^2} \right) \bar{v}(p, h)v(p, h) \right] \)
- There are other genuine expressions ...
- Translate Feynman diagrams into "helicity amplitudes": complex-valued functions of momenta & helicities.
- Spin-correlations etc. nearly come for free.
Taming the factorial growth

- In the helicity method
  - Reusing pieces: Calculate only once!
  - Factoring out: Reduce number of multiplications!

Implemented as a-posteriori manipulations of amplitudes.

- Better method: Recursion relations (recycling built in).
  Best candidate so far: Off-shell recursions

(Dyson-Schwinger, Berends-Giele etc.)
Efficient phase space integration

("Amateurs study strategy, professionals study logistics")

- Democratic, process-blind integration methods:
  - Rambo/Mambo: Flat & isotropic
  - HAAG/Sarge: Follows QCD antenna pattern

- Multi-channeling: Each Feynman diagram related to a phase space mapping (\(= \text{"channel"}\)), optimise their relative weights.

- Main problem: practical only up to \(O(10^k)\) channels.

- Some improvement by building phase space mappings recursively: More channels feasible, efficiency drops a bit.
I. Lowest order,
\[ \mathcal{O}(\alpha_{\text{em}}) : \]
\[ q\bar{q} \rightarrow Z^0 \]
Next-to-leading order (NLO) graphs

I. Lowest order,
\[ \mathcal{O}(\alpha_{em}) : \]
\[ q\bar{q} \rightarrow Z^0 \]

II. First-order real,
\[ \mathcal{O}(\alpha_{em}\alpha_s) : \]
\[ q\bar{q} \rightarrow Z^0 g \text{ etc.} \]
Next-to-leading order (NLO) graphs

I. Lowest order, \( \mathcal{O}(\alpha_{\text{em}}) \):
   \( q\bar{q} \rightarrow Z^0 \)

II. First-order real, \( \mathcal{O}(\alpha_{\text{em}}\alpha_s) \):
   \( q\bar{q} \rightarrow Z^0 g \) etc.

III. First-order virtual, \( \mathcal{O}(\alpha_{\text{em}}\alpha_s) \):
   \( q\bar{q} \rightarrow Z^0 \) with loops

\[ \frac{d\sigma}{dp_{\perp}} \]

lowest order finite \( \sigma_0 \)

real, \(+\infty\)

virtual, \(-\infty\)
\[ \sigma_{\text{NLO}} = \int_n d\sigma_{\text{LO}} + \int_{n+1} d\sigma_{\text{Real}} + \int_n d\sigma_{\text{Virt}} \]

Simple one-dimensional example: \( x \sim p_\perp / p_\perp \text{max}, \ 0 \leq x \leq 1 \)

Divergences regularized by \( d = 4 - 2\epsilon \) dimensions, \( \epsilon < 0 \)

\[ \sigma_{R+V} = \int_0^1 \frac{dx}{x^{1+\epsilon}} M(x) + \frac{1}{\epsilon} M_0 \]

KLN cancellation theorem: \( M(0) = M_0 \)
\[ \sigma_{\text{NLO}} = \int_{n} d\sigma_{\text{LO}} + \int_{n+1} d\sigma_{\text{Real}} + \int_{n} d\sigma_{\text{Virt}} \]

Simple one-dimensional example: \( x \sim p_{\perp}/p_{\perp \text{max}}, \ 0 \leq x \leq 1 \)
Divergences regularized by \( d = 4 - 2\epsilon \) dimensions, \( \epsilon < 0 \)

\[ \sigma_{R+V} = \int_{0}^{1} \frac{dx}{x^{1+\epsilon}} M(x) + \frac{1}{\epsilon} M_0 \]

KLN cancellation theorem: \( M(0) = M_0 \)

**Phase Space Slicing:**
Introduce arbitrary *finite* cutoff \( \delta \ll 1 \) (so \( \delta \gg |\epsilon| \))

\[ \sigma_{R+V} = \int_{\delta}^{1} \frac{dx}{x^{1+\epsilon}} M(x) + \int_{0}^{\delta} \frac{dx}{x^{1+\epsilon}} M(x) + \frac{1}{\epsilon} M_0 \]
\[ \approx \int_{\delta}^{1} \frac{dx}{x} M(x) + \int_{0}^{\delta} \frac{dx}{x^{1+\epsilon}} M_0 + \frac{1}{\epsilon} M_0 \]
\[ = \int_{\delta}^{1} \frac{dx}{x} M(x) + \frac{1}{\epsilon} (1 - \delta^{-\epsilon}) M_0 \approx \int_{\delta}^{1} \frac{dx}{x} M(x) + \ln \delta M_0 \]
NLO calculations – 2

Alternatively **Subtraction:**

\[
\sigma_{R+V} = \int_0^1 \frac{dx}{x^{1+\epsilon}} M(x) - \int_0^1 \frac{dx}{x^{1+\epsilon}} M_0 + \int_0^1 \frac{dx}{x^{1+\epsilon}} M_0 + \frac{1}{\epsilon} M_0
\]

\[
= \int_0^1 \frac{M(x) - M_0}{x^{1+\epsilon}} dx + \left(-\frac{1}{\epsilon} + \frac{1}{\epsilon}\right) M_0
\]

\[
\approx \int_0^1 \frac{M(x) - M_0}{x} dx + \mathcal{O}(1) M_0
\]
Alternatively **Subtraction**:

\[
\sigma_{R+V} = \int_0^1 \frac{dx}{x^{1+\epsilon}} M(x) - \int_0^1 \frac{dx}{x^{1+\epsilon}} M_0 + \int_0^1 \frac{dx}{x^{1+\epsilon}} M_0 + \frac{1}{\epsilon} M_0 \\
= \int_0^1 \frac{M(x) - M_0}{x^{1+\epsilon}} dx + \left( -\frac{1}{\epsilon} + \frac{1}{\epsilon} \right) M_0 \\
\approx \int_0^1 \frac{M(x) - M_0}{x} dx + \mathcal{O}(1) M_0
\]

NLO provides a more accurate answer for an integrated cross section:

**Warning!**

Neither approach operates with positive definite quantities. No obvious event-generator implementation. No trivial connection to physical events.
Scale choices

Cross section depends on factorization scale $\mu_F$ and renormalization scale $\mu_R$:

$$\sigma^{AB} = \sum_{i,j} \int \int d\mathbf{x}_1 d\mathbf{x}_2 f_i^{(A)}(\mathbf{x}_1, \mu_F) f_j^{(B)}(\mathbf{x}_2, \mu_F) \hat{\sigma}_{ij}(\alpha_s(\mu_R, \mu_F, \mu_R))$$

Historically common to put $Q = \mu_F = \mu_R$ but nowadays varied independently to gauge uncertainty of cross section prediction.

Typical variation factor $2^{\pm 1}$ around “natural value”, but beware
Availability of exact calculations (hadron colliders)

- Fixed order matrix elements ("parton level") are exact to a given perturbative order.
- Important to understand limitations:
  Only tree-level fully automated, 1-loop-level ongoing.

(and often quite a pain!)

---

Diagram:
- m loops
- n legs
- done
- for some processes
- first solutions
One Feynman graph can correspond to several possible colour flows, e.g. for $qg \rightarrow qg$:

while other $qg \rightarrow qg$ graphs only admit one colour flow:
so nontrivial mix of kinematics variables \((\hat{s}, \hat{t})\)
and colour flow topologies I, II:

\[
|A(\hat{s}, \hat{t})|^2 = |A_I(\hat{s}, \hat{t}) + A_{II}(\hat{s}, \hat{t})|^2 \\
= |A_I(\hat{s}, \hat{t})|^2 + |A_{II}(\hat{s}, \hat{t})|^2 + 2 \Re(A_I(\hat{s}, \hat{t})A_{II}^*(\hat{s}, \hat{t}))
\]

with \(\Re(A_I(\hat{s}, \hat{t})A_{II}^*(\hat{s}, \hat{t})) \neq 0\)

⇒ indeterminate colour flow, while

- showers *should* know it (coherence),
- hadronization *must* know it (hadrons singlets).

Normal solution:

\[
\frac{\text{interference}}{\text{total}} \propto \frac{1}{N_C^2 - 1}
\]

so split I : II according to proportions in the \(N_C \to \infty\) limit, i.e.

\[
|A(\hat{s}, \hat{t})|^2 = |A_I(\hat{s}, \hat{t})|^2_{\text{mod}} + |A_{II}(\hat{s}, \hat{t})|^2_{\text{mod}} \\
|A_{I(II)}(\hat{s}, \hat{t})|^2_{\text{mod}} = |A_I(\hat{s}, \hat{t}) + A_{II}(\hat{s}, \hat{t})|^2 \left(\frac{|A_{II}(\hat{s}, \hat{t})|^2}{|A_I(\hat{s}, \hat{t})|^2 + |A_{II}(\hat{s}, \hat{t})|^2}\right)_{N_C \to \infty}
\]