Introduction to Event Generators 1

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CTEQ/MCnet School, DESY, 10 July 2016

Course Plan and Position

Event generators: model and understand (LHC) events

Complementary to the "textbook" picture of particle physics, since event generators are close to how things work "in real life".

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Lecture 2	Parton showers: final and initial
Lecture 3	Multiparton interactions, other soft physics
Lecture 4	Hadronization, generator news, conclusions

+ 2 lectures on "Matching and merging" by Simon Plätzer

+ 3 hands-on tutorials with event generators

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Learn more:

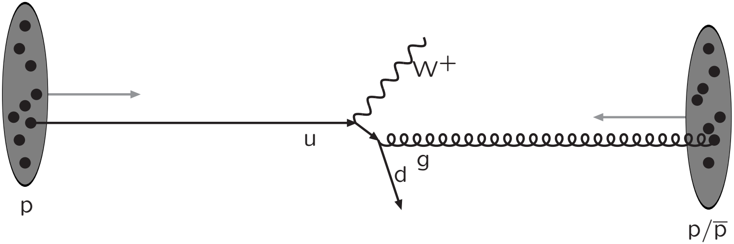
A. Buckley et al., "General-purpose event generators for LHC physics", Phys. Rep. 504 (2011) 145 [arXiv:1101.2599[hep-ph]]

The structure of an event – 1

Warning: schematic only, everything simplified, nothing to scale, . . .

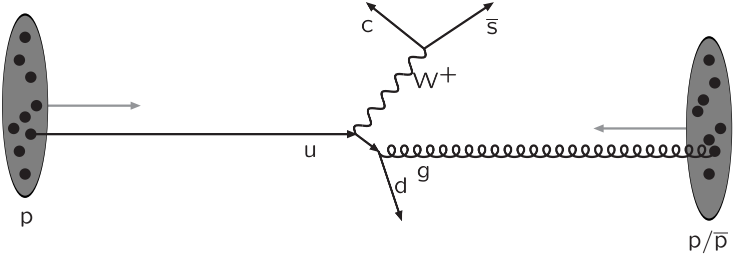
Incoming beams: parton densities

The structure of an event – 2



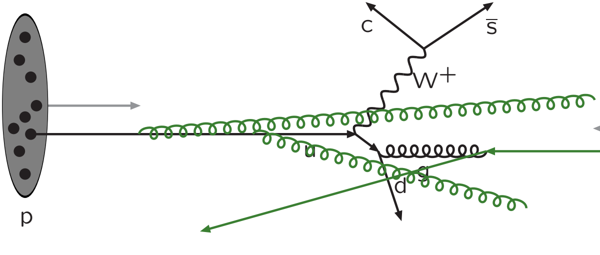
Hard subprocess: described by matrix elements

The structure of an event – 3



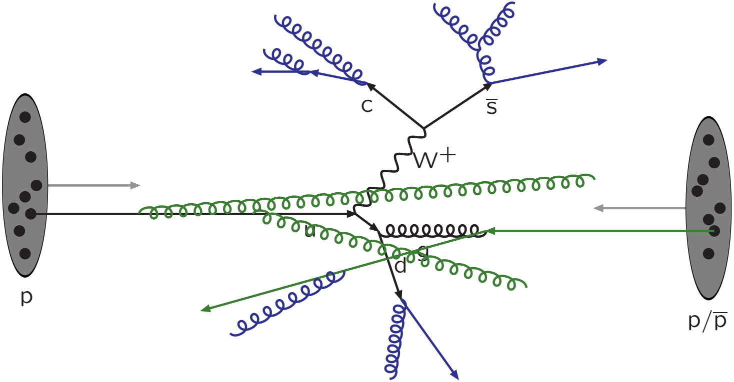
Resonance decays: correlated with hard subprocess

The structure of an event – 4



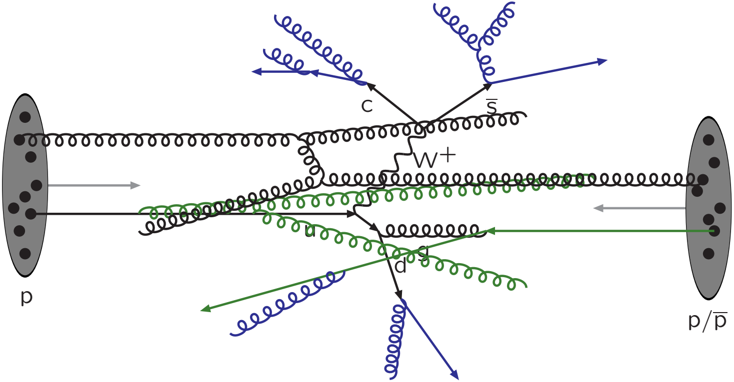
Initial-state radiation: spacelike parton showers

The structure of an event – 5



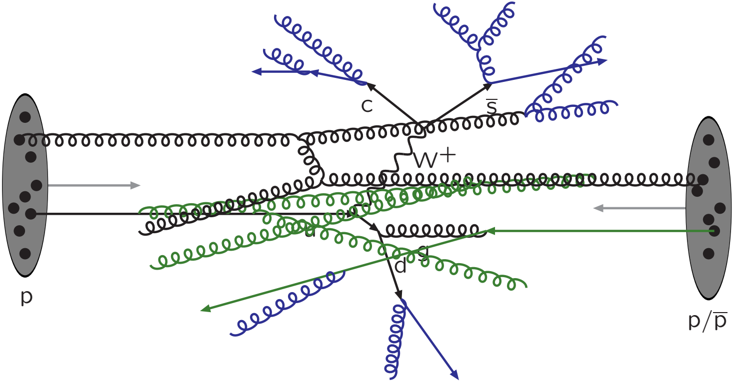
Final-state radiation: timelike parton showers

The structure of an event – 6



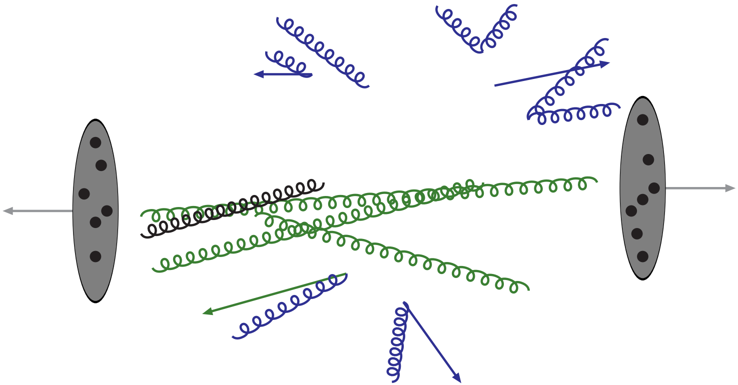
Multiple parton–parton interactions . . .

The structure of an event – 7



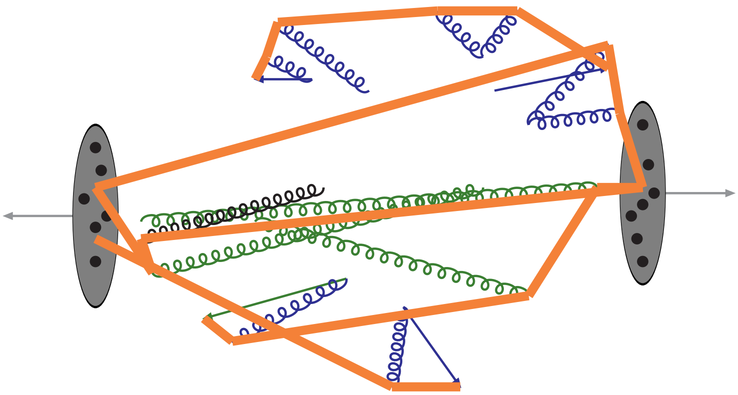
. . . with its initial- and final-state radiation

The structure of an event – 8



Beam remnants and other outgoing partons

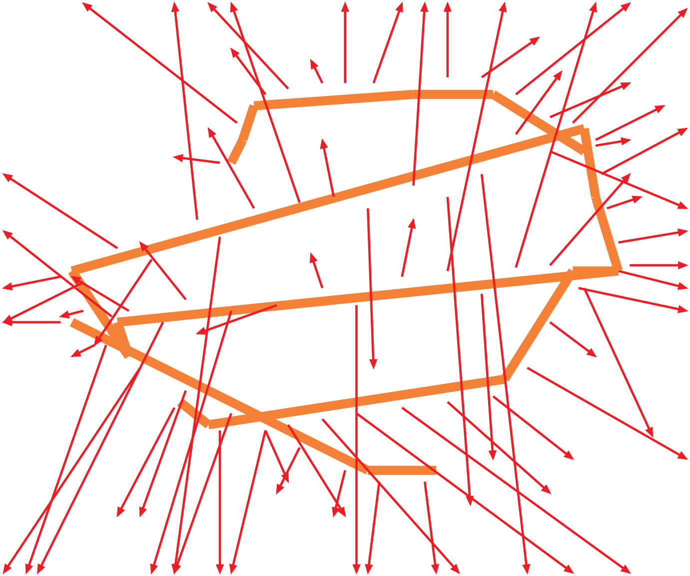
The structure of an event – 9



Everything is connected by colour confinement strings

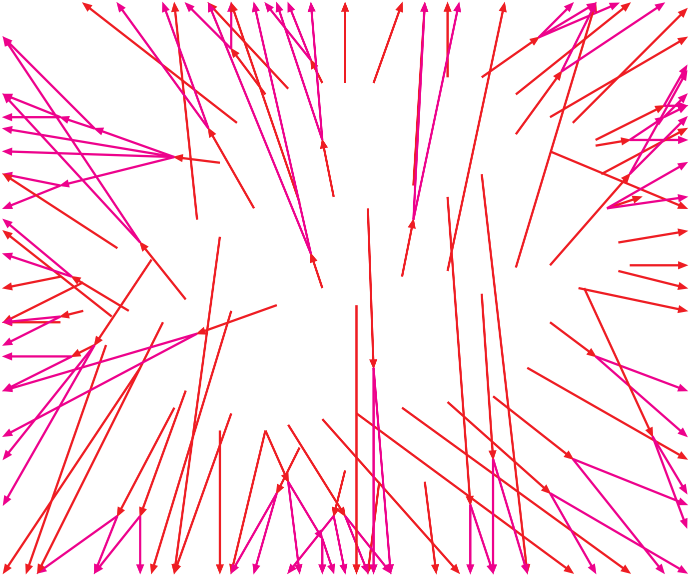
Recall! Not to scale: strings are of hadronic widths

The structure of an event – 10



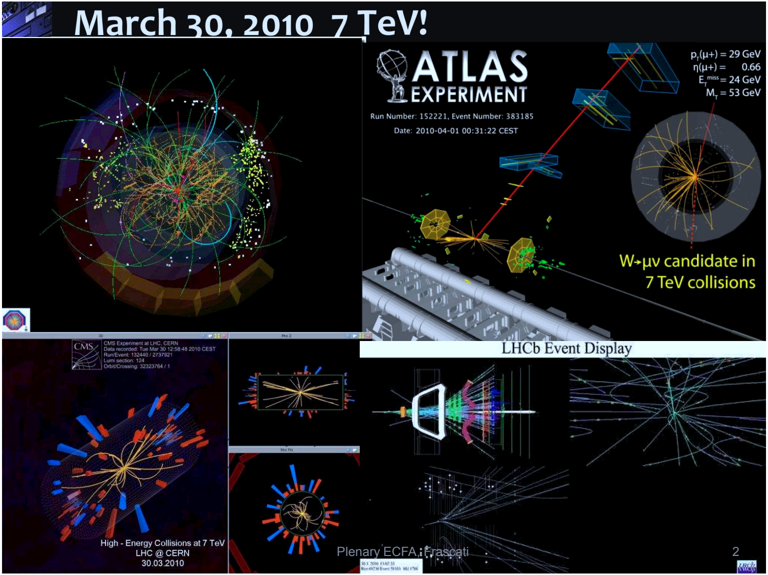
The strings fragment to produce primary hadrons

The structure of an event – 11



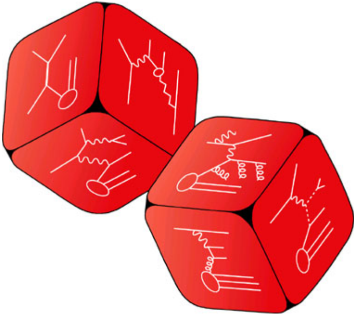
Many hadrons are unstable and decay further

The structure of an event – 12



These are the particles that hit the detector

A tour to Monte Carlo



. . . because Einstein was wrong: God does throw dice!
Quantum mechanics: amplitudes $\Longrightarrow$ probabilities
Anything that possibly can happen, will! (but more or less often)

Event generators: trace evolution of event structure.
Random numbers $\approx$ quantum mechanical choices.

The Monte Carlo method

Want to generate events in as much detail as Mother Nature
$\Longrightarrow$ get average *and* fluctutations right
$\Longrightarrow$ make random choices, $\sim$ as in nature

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$$\sigma_{\text{final state}} = \sigma_{\text{hard process}} \mathcal{P}_{\text{tot,hard process}\to\text{final state}}$$

(appropriately summed & integrated over non-distinguished final states)

where $\mathcal{P}_{\text{tot}} = \mathcal{P}_{\text{res}} \mathcal{P}_{\text{ISR}} \mathcal{P}_{\text{FSR}} \mathcal{P}_{\text{MPI}} \mathcal{P}_{\text{remnants}} \mathcal{P}_{\text{hadronization}} \mathcal{P}_{\text{decays}}$

with $\mathcal{P}_i = \prod_j \mathcal{P}_{ij} = \prod_j \prod_k \mathcal{P}_{ijk} = \ldots$ in its turn

$\Longrightarrow$ **divide and conquer**

The Monte Carlo method

Want to generate events in as much detail as Mother Nature
$\Longrightarrow$ get average *and* fluctutations right
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$\Longrightarrow$ **divide and conquer**

an event with n particles involves $\mathcal{O}(10n)$ random choices,
(flavour, mass, momentum, spin, production vertex, lifetime, ...)
LHC: $\sim$ 100 charged and $\sim$ 200 neutral (+ intermediate stages)
$\Longrightarrow$ several thousand choices
(of $\mathcal{O}(100)$ different kinds)

Why generators?

- Allow theoretical and experimental studies of *complex* multiparticle physics
- Large flexibility in physical quantities that can be addressed
- Vehicle of ideology to disseminate ideas from theorists to experimentalists

Can be used to

- predict event rates and topologies
  $\Rightarrow$ can estimate feasibility
- simulate possible backgrounds
  $\Rightarrow$ can devise analysis strategies
- study detector requirements
  $\Rightarrow$ can optimize detector/trigger design
- study detector imperfections
  $\Rightarrow$ can evaluate acceptance corrections

The workhorses: what are the differences?

Herwig, PYTHIA and Sherpa offer convenient frameworks
for LHC physics studies, covering all aspects above,
but with slightly different history/emphasis:

PYTHIA (successor to JETSET, begun in 1978):
originated in hadronization studies,
still special interest in soft physics.

Herwig (successor to EARWIG, begun in 1984):
originated in coherent showers (angular ordering),
cluster hadronization as simple complement.

Sherpa (APACIC++/AMEGIC++, begun in 2000):
had own matrix-element calculator/generator
originated with matching & merging issues.

MCnet

Herwig
PYTHIA
Sherpa
MadGraph

Plugin:
Ariadne
DIPSY
HEJ

CEDAR:
Rivet
Professor
HepForge
LHAPDF
HepMC

- EU-funded 2007–10, 2013–16, **2017–20**
- Generator development
- Services to community
- PhD student training
- Common activities
- Summer schools
  2016: DESY (w. CTEQ)
  2017: Lund, 3 - 7 July
- Short-term studentships (3 - 6 months).
  Formulate your project!
  Experimentalists welcome!

Nodes:
Manchester
CERN
Durham
Glasgow
Göttingen
Heidelberg
Karlsruhe
UC London
Louvain
Lund

Monash (Au)
SLAC (US)

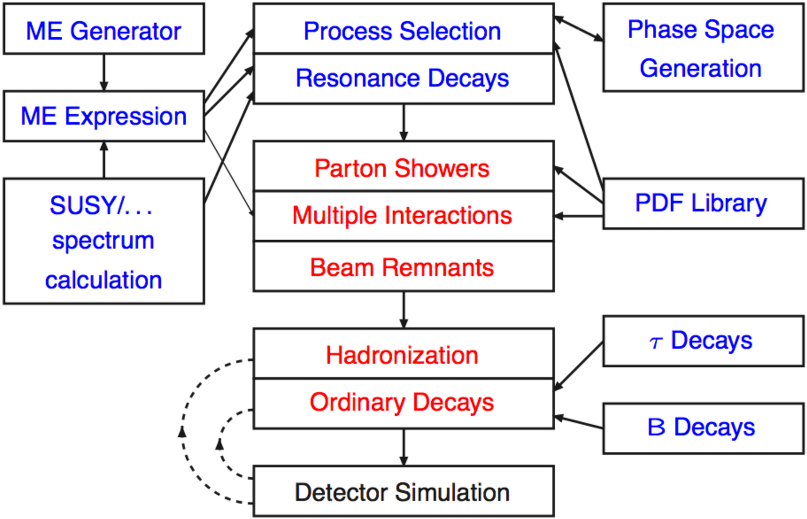
Other Relevant Software

Some examples (with apologies for many omissions):

- Other event/shower generators: PhoJet, Ariadne, Dipsy, Cascade, Vincia
- Matrix-element generators: MadGraph_aMC@NLO, Sherpa, Helac, Whizard, CompHep, CalcHep, GoSam
- Matrix element libraries: AlpGen, POWHEG BOX, MCFM, NLOjet++, VBFNLO, BlackHat, Rocket
- Special BSM scenarios: Prospino, Charybdis, TrueNoir
- Mass spectra and decays: SOFTSUSY, SPHENO, HDecay, SDecay
- Feynman rule generators: FeynRules
- PDF libraries: LHAPDF
- Resummed ($p_{\perp}$) spectra: ResBos
- Approximate loops: LoopSim
- Jet finders: anti-$k_{\perp}$ and FastJet
- Analysis packages: Rivet, Professor, MCPLOTS
- Detector simulation: GEANT, Delphes
- Constraints (from cosmology etc): DarkSUSY, MicrOmegas
- Standards: PDG identity codes, LHA, LHEF, SLHA, Binoth LHA, HepMC

Can be meaningfully combined and used for LHC physics!

Putting it together



Standardized interfaces essential!

PDG particle codes

A. Fundamental objects

1	d	11	e^-	21	g	32	Z'^0	39	G
2	u	12	ν_e	22	γ	33	Z''^0	41	R^0
3	s	13	μ^-	23	Z^0	34	W'^+	42	LQ
4	c	14	ν_μ	24	W^+	35	H^0	51	DM_0
5	b	15	τ^-	25	h^0	36	A^0		
6	t	16	ν_τ			37	H^+	...	...

add − sign for antiparticle, where appropriate

+ diquarks, SUSY, technicolor, ...

B. Mesons

$100\,|q_1| + 10\,|q_2| + (2s+1)$ with $|q_1| \geq |q_2|$

particle if heaviest quark u, $\bar{s}$, c, $\bar{b}$; else antiparticle

111	π^0	311	K^0	130	K^0_L	221	η^0	411	D^+	431	D^+_s
211	π^+	321	K^+	310	K^0_S	331	η'^0	421	D^0	443	J/ψ

C. Baryons

$1000\,q_1 + 100\,q_2 + 10\,q_3 + (2s+1)$

with $q_1 \geq q_2 \geq q_3$, or Λ-like $q_1 \geq q_3 \geq q_2$

2112	n	3122	Λ^0	2224	Δ^{++}	3214	Σ^{*0}
2212	p	3212	Σ^0	1114	Δ^-	3334	Ω^-

Les Houches LHA/LHEF event record

At initialization:

- beam kinds and E's
- PDF sets selected
- weighting strategy
- number of processes

Per process in initialization:

- integrated σ
- error on σ
- maximum $\mathrm{d}\sigma/\mathrm{d}(\mathrm{PS})$
- process label

Per event:

- number of particles
- process type
- event weight
- process scale
- α_{em}
- α_{s}
- (PDF information)

Per particle in event:

- PDG particle code
- status (decayed?)
- 2 mother indices
- colour & anticolour indices
- $(\mathrm{p}_x, p_y, p_z, E), m$
- lifetime τ
- spin/polarization

Detour: Monte Carlo techniques

"Spatial" problems: no memory/ordering

1. Integrate a function
2. Pick a point at random according to a probability distribution

"Temporal" problems: has memory

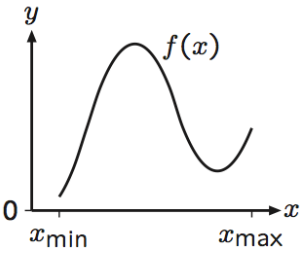
1. Radioactive decay: probability for a radioactive nucleus to decay at time t, given that it was created at time 0

In reality combined into multidimensional problems:

1. Random walk (variable step length and direction)
2. Charged particle propagation through matter (stepwise loss of energy by a set of processes)
3. **Parton showers** (cascade of successive branchings)
4. Multiparticle interactions (ordered multiple subcollisions)

Integration and selection

Assume function $f(x)$,
studied range $x_{\mathrm{min}} < x < x_{\mathrm{max}}$,
where $f(x) \geq 0$ everywhere



Two connected standard tasks:

1 Calculate (approximatively)

$$\int_{x_{\mathrm{min}}}^{x_{\mathrm{max}}} f(x')\,\mathrm{d}x'$$

2 Select x at random according to $f(x)$

In step 2 $f(x)$ is viewed as "probability distribution"
with implicit normalization to unit area,
and then step 1 provides overall correct normalization.

Integral as an area/volume

Theorem

An n-dimensional integration $\equiv$ an $n+1$-dimensional volume

$$\int f(x_1, \ldots, x_n) \, \mathrm{d}x_1 \ldots \mathrm{d}x_n \equiv \int \int_0^{f(x_1,\ldots,x_n)} 1 \, \mathrm{d}x_1 \ldots \mathrm{d}x_n \, \mathrm{d}x_{n+1}$$

since $\int_0^{f(x)} 1 \, \mathrm{d}y = f(x)$.

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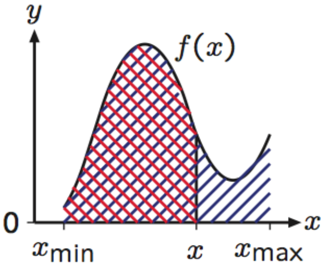
since $\int_0^{f(x)} 1\,\mathrm{d}y = f(x)$.
So, for 1 + 1 dimension, selection of x according to $f(x)$ is equivalent to uniform selection of (x, y) in the area $x_{\mathrm{min}} < x < x_{\mathrm{max}},\ 0 < y < f(x)$.
Therefore

$$\int_{x_{\mathrm{min}}}^{x} f(x')\,\mathrm{d}x' = R\int_{x_{\mathrm{min}}}^{x_{\mathrm{max}}} f(x')\,\mathrm{d}x'$$

(area to left of selected x is uniformly distributed fraction of whole area)



Analytical solution

If **know primitive function** $F(x)$ and **know inverse** $F^{-1}(y)$ then

$$F(x) - F(x_{\mathrm{min}}) = R\,(F(x_{\mathrm{max}}) - F(x_{\mathrm{min}})) = R\,A_{\mathrm{tot}}$$
$$\Longrightarrow x = F^{-1}(F(x_{\mathrm{min}}) + R\,A_{\mathrm{tot}})$$

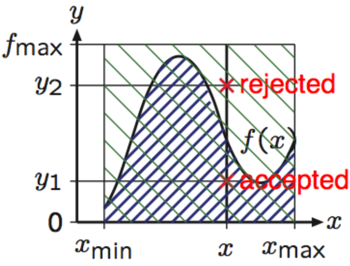
Proof: introduce $z = F(x_{\mathrm{min}}) + R\,A_{\mathrm{tot}}$. Then

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}x} = \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}R}\,\frac{\mathrm{d}R}{\mathrm{d}x} = 1\,\frac{1}{\frac{\mathrm{d}x}{\mathrm{d}R}} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}z}\,\frac{\mathrm{d}z}{\mathrm{d}R}} = \frac{1}{\frac{\mathrm{d}F^{-1}(z)}{\mathrm{d}z}\,\frac{\mathrm{d}z}{\mathrm{d}R}} = \frac{\frac{\mathrm{d}F(x)}{\mathrm{d}x}}{\frac{\mathrm{d}z}{\mathrm{d}R}} = \frac{f(x)}{A_{\mathrm{tot}}}$$

Hit-and-miss solution

If $f(x) \leq f_{\max}$ in $x_{\min} < x < x_{\max}$
use interpretation as an area

1. select
   $x = x_{\min} + R\,(x_{\max} - x_{\min})$
2. select $y = R\, f_{\max}$ (new R!)
3. while $y > f(x)$ cycle to 1



Integral as by-product:

$$I = \int_{x_{\min}}^{x_{\max}} f(x)\,\mathrm{d}x = f_{\max}\,(x_{\max} - x_{\min})\,\frac{N_{\mathrm{acc}}}{N_{\mathrm{try}}} = A_{\mathrm{tot}}\,\frac{N_{\mathrm{acc}}}{N_{\mathrm{try}}}$$

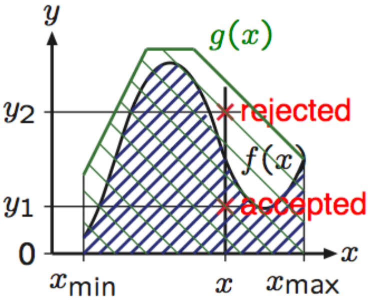
Binomial distribution with $p = N_{\mathrm{acc}}/N_{\mathrm{try}}$ and $q = N_{\mathrm{fail}}/N_{\mathrm{try}}$,
so error

$$\frac{\delta I}{I} = \frac{A_{\mathrm{tot}}\,\sqrt{p\,q/N_{\mathrm{try}}}}{A_{\mathrm{tot}}\,p} = \sqrt{\frac{q}{p\,N_{\mathrm{try}}}} = \sqrt{\frac{q}{N_{\mathrm{acc}}}} < \frac{1}{\sqrt{N_{\mathrm{acc}}}}$$

Importance sampling

Improved version of hit-and-miss:
If $f(x) \leq g(x)$ in
$x_{\mathrm{min}} < x < x_{\mathrm{max}}$
and $G(x) = \int g(x')\,\mathrm{d}x'$ is simple
and $G^{-1}(y)$ is simple

1. select x according to $g(x)$ distribution
2. select $y = R\,g(x)$ (new R!)
3. while $y > f(x)$ cycle to 1



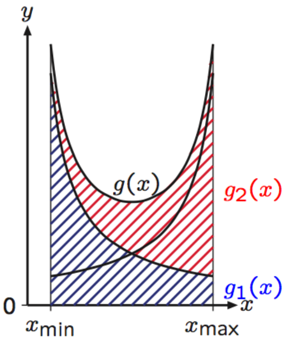
Multichannel

If $f(x) \leq g(x) = \sum_i g_i(x)$,
where all g_i "nice" ($G_i(x)$ invertible)
but $g(x)$ not

1. select i with relative probability

$$A_i = \int_{x_{\mathrm{min}}}^{x_{\mathrm{max}}} g_i(x')\,\mathrm{d}x'$$

2. select x according to $g_i(x)$
3. select $y = R\,g(x) = R\,\sum_i g_i(x)$
4. while $y > f(x)$ cycle to 1

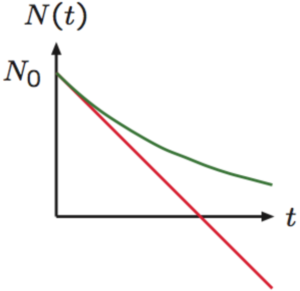


Works since

$$\int f(x)\,\mathrm{d}x = \int \frac{f(x)}{g(x)} \sum_i g_i(x)\,\mathrm{d}x = \sum_i A_i \int \frac{g_i(x)\,\mathrm{d}x}{A_i} \frac{f(x)}{g(x)}$$

Temporal methods: radioactive decays – 1

Consider "radioactive decay":
$N(t)$ = number of remaining nuclei at time t
but normalized to $N(0) = N_0 = 1$ instead, so equivalently
$N(t)$ = probability that (single) nucleus has not decayed by time t
$P(t) = -\mathrm{d}N(t)/\mathrm{d}t$ = probability for it to decay at time t



Naively $P(t) = c \Longrightarrow N(t) = 1 - ct$.
Wrong! Conservation of probability driven by depletion:
a given nucleus can only decay once

Correctly
$P(t) = cN(t) \Longrightarrow N(t) = \exp(-ct)$
i.e. exponential dampening
$P(t) = c\exp(-ct)$

There is memory in time!

Temporal methods: radioactive decays – 2

For radioactive decays $P(t) = cN(t)$, with c constant,
but now generalize to time-dependence:

$$P(t) = -\frac{\mathrm{d}N(t)}{\mathrm{d}t} = f(t)\,N(t)\ ; \quad f(t) \geq 0$$

Standard solution:

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -f(t)N(t) \quad \Longleftrightarrow \quad \frac{\mathrm{d}N}{N} = \mathrm{d}(\ln N) = -f(t)\,\mathrm{d}t$$

$$\ln N(t) - \ln N(0) = -\int_0^t f(t')\,\mathrm{d}t' \implies N(t) = \exp\left(-\int_0^t f(t')\,\mathrm{d}t'\right)$$

$$F(t) = \int^t f(t')\,\mathrm{d}t' \implies N(t) = \exp\left(-(F(t) - F(0))\right)$$

Assuming $F(\infty) = \infty$, i.e. always decay, sooner or later:

$$N(t) = R \implies t = F^{-1}(F(0) - \ln R)$$

The veto algorithm: problem

What now if $f(t)$ has no simple $F(t)$ or F^{-1}?
Hit-and-miss not good enough, since for $f(t) \leq g(t)$, g "nice",

$$t = G^{-1}(G(0) - \ln R) \implies N(t) = \exp\left(-\int_0^t g(t')\,\mathrm{d}t'\right)$$

$$P(t) = -\frac{\mathrm{d}N(t)}{\mathrm{d}t} = g(t)\,\exp\left(-\int_0^t g(t')\,\mathrm{d}t'\right)$$

and hit-or-miss provides rejection factor $f(t)/g(t)$, so that

$$P(t) = f(t)\,\exp\left(-\int_0^t g(t')\,\mathrm{d}t'\right)$$

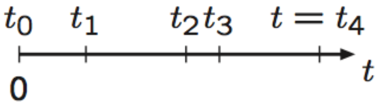
(modulo overall normalization), where it ought to have been

$$P(t) = f(t)\,\exp\left(-\int_0^t f(t')\,\mathrm{d}t'\right)$$

The veto algorithm: solution

The veto algorithm

1. start with $i = 0$ and $t_0 = 0$
2. $i = i + 1$
3. $t_i = G^{-1}(G(t_{i-1}) - \ln R)$, i.e $t_i > t_{i-1}$
4. $y = R\, g(t)$
5. while $y > f(t)$ cycle to 2



That is, when you fail, you keep on going from the time when you failed, and *do not* restart at time $t = 0$. (Memory!)

The veto algorithm: proof – 1

Study probability to have i intermediate failures before success:

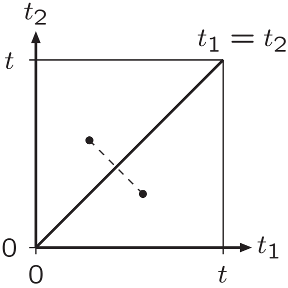
Define $S_g(t_a, t_b) = \exp\left(-\int_{t_a}^{t_b} g(t')\,\mathrm{d}t'\right)$ ("Sudakov factor")

$$P_0(t) = P(t = t_1) = g(t)\,S_g(0,t)\,\frac{f(t)}{g(t)} = f(t)\,S_g(0,t)$$

$$\begin{aligned} P_1(t) &= P(t = t_2) \\ &= \int_0^t \mathrm{d}t_1\, g(t_1) S_g(0,t_1)\left(1 - \frac{f(t_1)}{g(t_1)}\right) g(t)\,S_g(t_1,t)\,\frac{f(t)}{g(t)} \\ &= f(t)\,S_g(0,t) \int_0^t \mathrm{d}t_1\,(g(t_1) - f(t_1)) = P_0(t)\,I_{g-f} \end{aligned}$$

$$\begin{aligned} P_2(t) &= \cdots = P_0(t) \int_0^t \mathrm{d}t_1\,(g(t_1) - f(t_1)) \int_{t_1}^t \mathrm{d}t_2\,(g(t_2) - f(t_2)) \\ &= P_0(t) \int_0^t \mathrm{d}t_1\,(g(t_1) - f(t_1)) \int_0^t \mathrm{d}t_2\,(g(t_2) - f(t_2))\,\theta(t_2 - t_1) \\ &= P_0(t)\,\frac{1}{2}\left(\int_0^t \mathrm{d}t_1\,(g(t_1) - f(t_1))\right)^2 = P_0(t)\,\frac{1}{2}\,I_{g-f}^2 \end{aligned}$$

The veto algorithm: proof – 2



Generally, i intermediate times corresponds to $i!$ equivalent ordering regions.

$$P_i(t) = P_0(t)\,\frac{1}{i!}\,I_{g-f}^i$$

$$\begin{aligned} P(t) &= \sum_{i=0}^{\infty} P_i(t) = P_0(t)\sum_{i=0}^{\infty}\frac{I_{g-f}^i}{i!} = P_0(t)\exp(I_{g-f}) \\ &= f(t)\,\exp\left(-\int_0^t g(t')\,\mathrm{d}t'\right)\,\exp\left(\int_0^t (g(t') - f(t'))\,\mathrm{d}t'\right) \\ &= f(t)\,\exp\left(-\int_0^t f(t')\,\mathrm{d}t'\right) \end{aligned}$$

The winner takes it all

Assume "radioactive decay" with two possible decay channels 1&2

$$P(t) = -\frac{\mathrm{d}N(t)}{\mathrm{d}t} = f_1(t)N(t) + f_2(t)N(t)$$

Alternative 1:
use normal veto algorithm with $f(t) = f_1(t) + f_2(t)$.
Once t selected, pick decays 1 or 2 in proportions $f_1(t) : f_2(t)$.

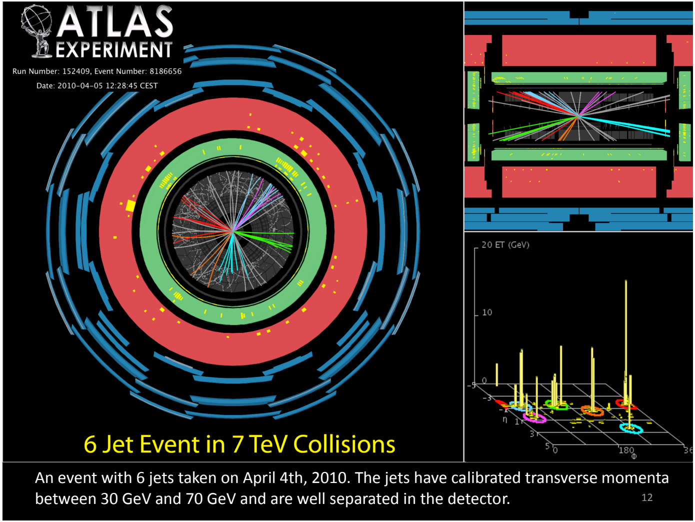
Alternative 2:

The winner takes it all

select t_1 according to $P_1(t_1) = f_1(t_1)N_1(t_1)$
and t_2 according to $P_2(t_2) = f_2(t_2)N_2(t_2)$,
i.e. as if the other channel did not exist.
If $t_1 < t_2$ then pick decay 1, while if $t_2 < t_1$ pick decay 2.

Equivalent by simple proof.

Multijets – the need for Higher Orders



$2 \rightarrow 6$ process or $2 \rightarrow 2$ dressed up by bremsstrahlung!?

Perturbative QCD

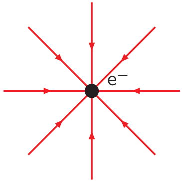
Perturbative calculations ⇒ **Matrix Elements**.
Improved calculational techniques allows
⋆ more **legs** (= final-state partons)
⋆ more **loops** (= virtual partons not visible in final state)
but with limitations, especially for loops.

Parton Showers:
approximations to matrix element behaviour,
most relevant for multiple emissions at low energies and/or angles.
To be described next.

Matching and Merging:
methods to combine matrix elements (at high scales)
with parton showers (at low scales),
with a consistent and smooth transition.
To be covered in lectures by Simon Plätzer.

In the beginning: Electrodynamics

An electrical charge, say an electron, is surrounded by a field:



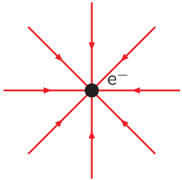
In the beginning: Electrodynamics

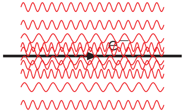
An electrical charge, say an electron,
is surrounded by a field:

For a rapidly moving charge
this field can be expressed in terms of
an equivalent flux of photons:

$$\mathrm{dn}_\gamma \approx \frac{2\alpha_{\mathrm{em}}}{\pi} \frac{\mathrm{d}\theta}{\theta} \frac{\mathrm{d}\omega}{\omega}$$

Equivalent Photon Approximation,
or method of virtual quanta (e.g. Jackson)
(Bohr; Fermi; Weiszäcker, Williams ∼1934)





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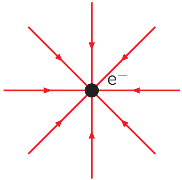

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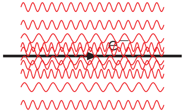

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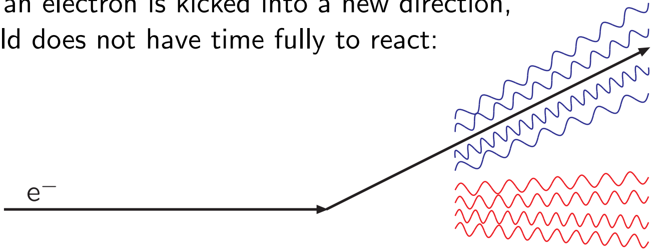
θ: collinear divergence, saved by $m_{\mathrm{e}} > 0$ in full expression.

ω: true divergence, $n_\gamma \propto \int \mathrm{d}\omega/\omega = \infty$, but $E_\gamma \propto \int \omega \, \mathrm{d}\omega/\omega$ finite.

These are virtual photons: continuously emitted and reabsorbed.

In the beginning: Bremsstrahlung

When an electron is kicked into a new direction,
the field does not have time fully to react:



- Initial State Radiation (ISR):
  part of it continues $\sim$ in original direction of e
- Final State Radiation (FSR):
  the field needs to be regenerated around outgoing e,
  and transients are emitted $\sim$ around outgoing e direction

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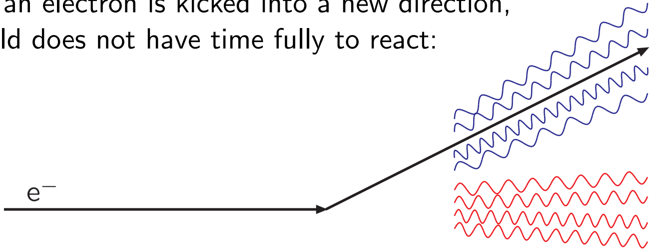

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Emission rate provided by equivalent photon flux in both cases.
Approximate cutoffs related to timescale of process:
the more violent the hard collision, the more radiation!

In the beginning: Exponentiation

Assume $\sum E_\gamma \ll E_\mathrm{e}$ such that energy-momentum conservation is not an issue. Then

$$\mathrm{d}\mathcal{P}_\gamma = \mathrm{dn}_\gamma \approx \frac{2\alpha_\mathrm{em}}{\pi} \frac{\mathrm{d}\theta}{\theta} \frac{\mathrm{d}\omega}{\omega}$$

is the probability to find a photon at ω and θ,
irrespectively of which other photons are present.

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Uncorrelated $\Rightarrow$ Poissonian number distribution:

$$\mathcal{P}_i = \frac{\langle n_\gamma \rangle^i}{i!} \, e^{-\langle n_\gamma \rangle}$$

with

$$\langle n_\gamma \rangle = \int_{\theta_\mathrm{min}}^{\theta_\mathrm{max}} \int_{\omega_\mathrm{min}}^{\omega_\mathrm{max}} \mathrm{dn}_\gamma \approx \frac{2\alpha_\mathrm{em}}{\pi} \ln \left(\frac{\theta_\mathrm{max}}{\theta_\mathrm{min}} \right) \ln \left(\frac{\omega_\mathrm{max}}{\omega_\mathrm{min}} \right)$$

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Note that $\int \mathrm{d}\mathcal{P}_\gamma = \int \mathrm{dn}_\gamma > 1$ is not a problem:
proper interpretation is that *many* photons are emitted.

Exponentiation: reinterpretation of $\mathrm{d}\mathcal{P}_\gamma$ into Poissonian.

QED: Fixed Order Perturbation Theory

Order-by-order perturbative ME calculation contains fully differential distributions of multi-γ emissions,

but integrating the main contributions (leading logs) gives

$$
\begin{aligned}
\frac{\sigma_{0\gamma}}{\sigma_0} &\approx 1 - \alpha_{\mathrm{em}} N + \alpha_{\mathrm{em}}^2 \frac{N^2}{2} - \alpha_{\mathrm{em}}^3 \frac{N^3}{6} \\
\frac{\sigma_{1\gamma}}{\sigma_0} &\approx + \alpha_{\mathrm{em}} N - \alpha_{\mathrm{em}}^2 N^2 + \alpha_{\mathrm{em}}^3 \frac{N^3}{2} \\
\frac{\sigma_{2\gamma}}{\sigma_0} &\approx + \alpha_{\mathrm{em}}^2 \frac{N^2}{2} - \alpha_{\mathrm{em}}^3 \frac{N^3}{2} \\
\frac{\sigma_{3\gamma}}{\sigma_0} &\approx + \alpha_{\mathrm{em}}^3 \frac{N^3}{6}
\end{aligned}
$$

which is the expanded form of the Poissonian $\mathcal{P}_i = \langle n_\gamma \rangle^i \, e^{-\langle n_\gamma \rangle} / i!$ with $\langle n_\gamma \rangle = \alpha_{\mathrm{em}} N$.

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For practical applications two different regions

- large $\theta, \omega \Rightarrow$ rapidly convergent perturbation theory
- small $\theta, \omega \Rightarrow$ exponentiation needed, even if approximate

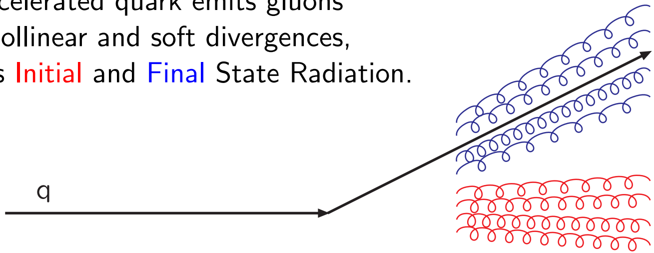
So how is QCD the same?

- A quark is surrounded by a gluon field

$$\mathrm{d}\mathcal{P}_{\mathrm{g}} = \mathrm{dn}_{\mathrm{g}} \approx \frac{8\alpha_{\mathrm{s}}}{3\pi} \frac{\mathrm{d}\theta}{\theta} \frac{\mathrm{d}\omega}{\omega}$$

  i.e. only differ by substitution $\alpha_{\mathrm{em}} \to 4\alpha_{\mathrm{s}}/3$.

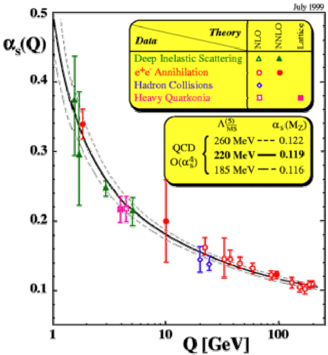
- An accelerated quark emits gluons with collinear and soft divergences, and as Initial and Final State Radiation.



- Typically $\langle n_{\mathrm{g}} \rangle = \int \mathrm{dn}_{\mathrm{g}} \gg 1$ since $\alpha_{\mathrm{s}} \gg \alpha_{\mathrm{em}}$
  $\Rightarrow$ even more pressing need for exponentiation.

So how is QCD different?

- QCD is non-Abelian, so a gluon is charged and is surrounded by its own field: emission rate $4\alpha_{\mathrm{s}}/3 \to 3\alpha_{\mathrm{s}}$, field structure more complicated, interference effects more important.
- $\alpha_s(Q^2)$ diverges for $Q^2 \to \Lambda^2_{\mathrm{QCD}}$, with $\Lambda_{\mathrm{QCD}} \sim 0.2\,\mathrm{GeV} = 1\,\mathrm{fm}^{-1}$.
- Confinement: gluons below Λ_{QCD} not resolved $\Rightarrow$ de facto cutoffs.



Unclear separation between "accelerated charge" and "emitted radiation": many possible Feynman graphs ≈ histories.